

Practical Mathematics

TDH Baber



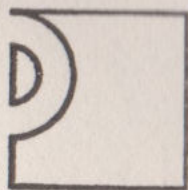
Definitions and
Formulae
for Students

Definitions and Formulae for Students: **PRACTICAL MATHEMATICS**

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Section 1

ALGEBRA

1.1. POWERS OR INDICES

If n is a *positive integer*,

$x^n = x \times x \times x \dots \times x$ a continued product of n factors

Otherwise, provided $x \neq 0$,

(a) $n = 0, x^0 = 1$

(b) $n = -p$, where p is real and positive, $x^{-p} = \frac{1}{x^p}$

(c) $n = p/q$, where p, q are both integers, $x^{p/q} = \sqrt[q]{x^p}$

If $x = 0, 0^n = 0$ for all values of n .

Laws of Indices

$$x^m \times x^n = x^{m+n} \quad x^m \div x^n = x^{m-n} \quad (x^m)^n = x^{mn}$$

1.2. LOGARITHMS

DEFINITION: If $x = a^c$, then $\log_a x = c$.

$$\log(xy) = \log x + \log y$$

$$\log(x/y) = \log x - \log y$$

$$\log(x^n) = n \log x$$

$$\log(\sqrt[n]{x}) = (1/n) \log x$$

$$\log_{10} x = \log_{10} e \times \log_e x = \log_e x / \log_e 10 = 0.4343 \log_e x$$

($e = 2.718 \dots$)

1.3. IMPORTANT FACTORS

$$x^2 - a^2 = (x + a)(x - a)$$

$$x^3 \pm a^3 = (x \pm a)(x^2 \mp ax + a^2)$$

$$x^3 + y^3 + z^3 - 3xyz \\ = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

1.4. QUADRATIC EQUATIONS

If the quadratic equation $ax^2 + bx + c = 0$ has roots α, β , then

$$x^2 + (b/a)x + c/a \equiv (x - \alpha)(x - \beta) \equiv x^2 - (\alpha + \beta)x + \alpha\beta$$

$$\text{Sum of roots} = \alpha + \beta = -b/a$$

$$\text{Product of roots} = \alpha\beta = c/a$$

$$\alpha, \beta = [-b \pm \sqrt{(b^2 - 4ac)}]/2a$$

$$\text{Discriminant } D = b^2 - 4ac$$

$$D > 0, \text{ roots real and unequal}$$

$$D = 0, \text{ roots real and equal}$$

$$D < 0, \text{ roots complex and unequal (complex quantities, p. 23)}$$

1.5. REMAINDER THEOREM

If n is a positive integer and $a_n, a_{n-1}, \dots, a_1, a_0$ are constants,

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

$f(x)$ is a *polynomial* of degree n .

If $f(x)$ is divided by $(x - c)$, the remainder is $f(c)$.

Thus $(x - c)$ is a factor of $f(x)$ if $f(c) = 0$.

1.6. RATIO AND PROPORTION

The ratio of a to b (written $a:b$) is denoted by the fraction a/b .

a, b, c and d are in proportion if $a/b = c/d$

If $a/b = c/d$ then

$$ad = bc \qquad \frac{a \pm b}{b} = \frac{c \pm d}{d}$$

$$\frac{a+b}{a-b} = \frac{c+d}{c-d} \qquad \frac{la+mb}{pa+qb} = \frac{lc+md}{pa+qd}$$

(where l, m, p, q are real numbers).

1.7. VARIATION

If k is a constant,

Direct variation: if $y \propto x$, then $y = kx$

Inverse variation: if $y \propto 1/x$, then $y = k/x$

If $z \propto x$ when y is constant, and if $z \propto y$ when x is constant then $z \propto xy$, i.e. $z = kxy$ when both x and y vary.

1.8. SIMPLE SERIES

An *Arithmetic Progression* is a series of numbers such that any term is derived from the previous term by *adding* to it a constant quantity, called the *common difference* (d).

Let a be the first term of the series, then the n th term is

$$l = a + (n - 1)d$$

and the sum of n terms is given by

$$\begin{aligned} S_n &= a + (a + d) + (a + 2d) + \dots + (a + \overline{n-1}d) \\ &= \frac{1}{2}n(2a + \overline{n-1}d) = \frac{1}{2}n(a + l) \end{aligned}$$

A *Geometric Progression* is a series of numbers such that any term is derived from the previous term by *multiplying* the latter by a constant quantity called the *common ratio* (r). The n th term is ar^{n-1} where a is the first term.

$$S_n = a + ar + ar^2 + \dots + ar^{n-1} = a(1 - r^n)/(1 - r)$$

The sum to infinity $= S_\infty = a/(1 - r)$ ($|r| < 1$)

Harmonic Progression. A series of numbers $a, b, c, \dots$ will be in Harmonic Progression if their inverses $1/a, 1/b, 1/c, \dots$ are in Arithmetic Progression so that

$$1/c - 1/b = 1/b - 1/a$$

Means of Two Numbers

$$\text{Arithmetic mean} = A = \frac{1}{2}(a + b)$$

$$\text{Geometric mean} = G = \sqrt{ab}$$

$$\text{Harmonic mean} = H = \frac{2ab}{a + b} = G^2/A$$

Sums of Important Series

$$\text{Sum of the first } n \text{ natural numbers} = \frac{1}{2}n(n + 1)$$

$$\text{Sum of the squares of the first } n \text{ natural numbers}$$

$$= \frac{1}{6}n(n + 1)(2n + 1)$$

$$\text{Sum of the cubes of the first } n \text{ natural numbers} = [\frac{1}{2}n(n + 1)]^2$$

1.9. COMBINATORIAL FORMULAE

Any arrangement of n objects in a definite order is called a *permutation*.

The number of permutations of n objects amongst themselves is

$$n(n - 1)(n - 2) \dots 2.1 = n!$$

The number of permutations of n objects r at a time is

$${}^nP_r = n!/(n - r)!$$

The number of *combinations* of n objects taken r at a time is the number of selections, each of r objects, without reference to their order within a selection, and this number is denoted by

$${}^nC_r \text{ or } \binom{n}{r}.$$

$$\binom{n}{r} = \frac{n!}{r!(n - r)!} = \frac{n(n - 1)(n - 2) \dots (n - r + 1)}{r!}$$

$$= \binom{n}{n - r}$$

The number of ways in which n objects may be divided into k groups containing respectively $n_1, n_2, n_3, \dots, n_k$ objects such that $n_1 + n_2 + \dots + n_k = n$ is

$$n!/(n_1! n_2! \dots n_k!)$$

1.10. BINOMIAL AND MULTINOMIAL THEOREMS

Binomial Theorem

$$(x + a)^n = \left\{ \binom{n}{0} x^n + \binom{n}{1} x^{n-1} a + \binom{n}{2} x^{n-2} a^2 + \dots \right. \\ \left. + \binom{n}{r} x^{n-r} a^r + \dots + \binom{n}{n} a^n \right\}$$

(n is a positive integer.) (See also Binomial Series, p. 44.)

The values of the binomial coefficients for different values of n are:

<i>Value of n</i>	<i>Binomial coefficients</i>											
1					1		1					
2					1		2		1			
3				1		3		3		1		
4			1		4		6		4		1	
5		1		5		10		10		5		1
etc.						etc.						

Note that the binomial coefficients for any value of n may be obtained by summing consecutive pairs of coefficients in the preceding line.

Multinomial Theorem

$$(x_1 + x_2 + \dots + x_k)^n \\ = \sum \frac{n!}{n_1! n_2! \dots n_k!} x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$$

where $n = \sum_{i=1}^k n_i$ and each n_i takes integral values from 0 to n .

1.11. DETERMINANTS

Second order determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ has the value $(a_1b_2 - a_2b_1)$

Third order determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ has the value $a_1A_1 + b_1B_1 + c_1C_1$

where $A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ $B_1 = \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}$ $C_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

The solutions of the simultaneous equations $a_1x + b_1y = c_1$
 $a_2x + b_2y = c_2$
 are given by

$$\frac{x}{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

and the solutions of $a_1x + b_1y + c_1z = d_1$ are given by
 $a_2x + b_2y + c_2z = d_2$
 $a_3x + b_3y + c_3z = d_3$

$$\frac{x}{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Section 2

MENSURATION AND TRIGONOMETRY

2.1. ANGLE

One *radian* is the angle subtended at the centre of a circle by an arc equal in length to the radius r .

$$2\pi \text{ radians} = 360^\circ$$

$$\pi/180 \text{ radians} = 1^\circ$$

$$1 \text{ radian} = 180^\circ/\pi = 57.3^\circ$$

$$\text{Length of arc} = s = r\theta \text{ } (\theta \text{ in radians})$$

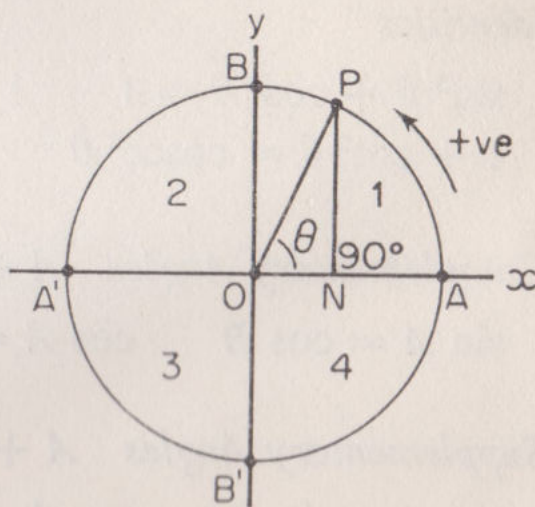
2.2. TRIGONOMETRICAL RATIOS

Let θ be the angle through which the radius OA would turn to reach the position OP , i.e. $\theta = \angle PON$.

(Convention: counter-clockwise rotation gives a positive angle and clockwise rotation gives a negative angle.)

$$\angle OPN = 90^\circ$$

ON and PN are directed lines in accordance with the convention



for Cartesian coordinates so that

ON is positive in the 1st and 4th quadrants
and negative in the 2nd and 3rd

NP is positive in the 1st and 2nd quadrants
and negative in the 3rd and 4th

DEFINITIONS

$$\sin \theta = \text{NP/OP} \quad \cos \theta = \text{ON/OP} \quad \tan \theta = \text{NP/ON}$$

Algebraic signs of the above ratios in the four quadrants

$$\begin{array}{c|c} + & + \\ \hline - & - \end{array}$$

sin

$$\begin{array}{c|c} - & + \\ \hline - & + \end{array}$$

cos

$$\begin{array}{c|c} - & + \\ \hline + & - \end{array}$$

tan

Further Definitions and Identities

$$\operatorname{cosec} \theta = 1/\sin \theta \quad \sec \theta = 1/\cos \theta \quad \cot \theta = 1/\tan \theta$$

$\sin^{-1}x$ is the angle θ ($-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$) such that $\sin \theta = x$

$\cos^{-1}x$ is the angle θ ($0 < \theta < \pi$) such that $\cos \theta = x$

$\tan^{-1}x$ is the angle θ ($-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$) such that $\tan \theta = x$

Identities

$$\sin^2 \theta + \cos^2 \theta = 1 \quad 1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \operatorname{cosec}^2 \theta \quad \tan \theta = \sin \theta / \cos \theta$$

Complementary Angles $A + B = 90^\circ$

$$\sin A = \cos B \quad \cos A = \sin B \quad \tan A = \cot B$$

Supplementary Angles $A + B = 180^\circ$

$$\sin A = \sin B \quad \cos A = -\cos B \quad \tan A = -\tan B$$

2.3. MENSURATION

Plane Areas

Parallelogram Sides a, b and included angle α . Perpendicular distance between parallel sides of length $a = h = b \sin \alpha$

$$\text{Area} = ah = ab \sin \alpha$$

Trapezium Area = $\frac{1}{2}$ (Sum of parallel sides) $\times$ Perpendicular distance between them

Circle Radius r , diameter $d = 2r$.

$$\text{Circumference} = 2\pi r = \pi d \quad \text{Area} = \pi r^2 = \frac{1}{4}\pi d^2$$

If an arc subtends an angle θ radians at the centre

$$\text{Area of sector} = \frac{1}{2}r^2\theta$$

$$\text{Area of segment} = \frac{1}{2}r^2(\theta - \sin \theta)$$

Ellipse Semi-axes a, b ; area = πab .

Solids

Notation: V = volume, C = curved surface area, S = total surface area.

Solid Circular Cylinder

h = height, R = radius

$$V = \pi R^2 h \quad C = 2\pi R h \quad S = 2\pi R h + 2\pi R^2$$

Hollow Circular Cylinder

r = inner radius

$$V = \pi(R^2 - r^2)h$$

Pyramid and Cone

$$V = \frac{1}{3} (\text{Area of base}) \times \text{Perpendicular height}$$

Right Circular Cone

h = height, R = radius of base, l = slant height = $\sqrt{R^2 + h^2}$

$$V = \frac{1}{3}\pi R^2 h \quad C = \pi Rl$$

$$S = \pi R^2 + \pi Rl$$

$$\text{Volume of frustum} = \frac{1}{3}\pi h (R^2 + Rr + r^2)$$

where R, r are radii of the ends and h is the height of the frustum.

$$\text{Sphere Radius } R \quad V = \frac{4}{3}\pi R^3 \quad C = 4\pi R^2$$

$$\text{Volume of a zone} = \frac{1}{8}\pi h [3(R^2 + r^2) + h^2]$$

where R, r are radii of the ends and h is the height of the zone.

2.4. SOLUTION OF TRIANGLES

Triangle ABC: sides a, b, c . Perpendicular height above the base a is h . Angle A is opposite side a , etc. R, r are the radii of the circumscribed and inscribed circles of triangle ABC.

$$\text{Semi-perimeter} = s = \frac{1}{2}(a + b + c)$$

$$\text{Area} = \frac{1}{2}ah = \frac{1}{2}ab \sin C = \sqrt{[s(s-a)(s-b)(s-c)]} = rs$$

$$\text{If } C = 90^\circ, c^2 = a^2 + b^2 \quad (\text{Pythagoras' Theorem})$$

$$\text{Sine formula: } a/\sin A = b/\sin B = c/\sin C = 2R$$

$$\text{Cosine formulae: } a^2 = b^2 + c^2 - 2bc \cos A \quad \text{etc.}$$
$$a = b \cos C + c \cos B \quad \text{etc.}$$

$$\sin \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}} \quad \cos \frac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}$$

$$\tan \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Area of a cyclic quadrilateral (sides a, b, c, d)

$$\sqrt{[(s-a)(s-b)(s-c)(s-d)]} \quad \text{where } s = \frac{1}{2}(a + b + c + d)$$

2.5. COMPOUND ANGLES

$$\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan (A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$$

$$\sin 3A = 3 \sin A - 4 \sin^3 A$$

$$\cos 3A = 4 \cos^3 A - 3 \cos A$$

$$\sin C + \sin D = 2 \sin \frac{1}{2}(C + D) \cos \frac{1}{2}(C - D)$$

$$\sin C - \sin D = 2 \cos \frac{1}{2}(C + D) \sin \frac{1}{2}(C - D)$$

$$\cos C + \cos D = 2 \cos \frac{1}{2}(C + D) \cos \frac{1}{2}(C - D)$$

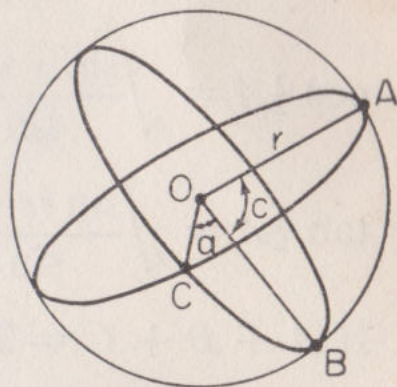
$$\cos C - \cos D = -2 \sin \frac{1}{2}(C + D) \sin \frac{1}{2}(C - D)$$

2.6. SPHERICAL TRIGONOMETRY

Spherical Triangles

The section of a sphere by a plane through its centre is called a *great circle*. A *spherical triangle* is the figure bounded by three great circles.

The sides BC, CA, AB of the spherical triangle ABC are measured by the angles a, b, c , which they subtend at the centre O. The angles A, B, C are the angles between the planes of the great circles.



$$\text{Spherical excess} = E = A + B + C - \pi$$

$$\text{Area of spherical triangle ABC} = (A + B + C - \pi)r^2 = Er^2$$

$$\text{Sine formula: } \sin A/\sin a = \sin B/\sin b = \sin C/\sin c$$

Cosine formulae

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \quad \text{etc.}$$

Cotangent formulae

$$\cot b \sin c = \cos c \cos A + \sin A \cot B \quad \text{etc.}$$

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C} \quad \text{etc.}$$

$$\tan \frac{1}{2}(A + B) = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \frac{1}{2}C$$

$$\tan \frac{1}{2}(A - B) = \frac{\sin \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \frac{1}{2}C$$

$$\tan \frac{1}{2}(a + b) = \frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} \tan \frac{1}{2}c$$

$$\tan \frac{1}{2}(a - b) = \frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} \tan \frac{1}{2}c$$

If $a + b + c = 2s$

$$\sin \frac{1}{2}A = \sqrt{\frac{\sin(s - b) \sin(s - c)}{\sin b \sin c}}$$

$$\cos \frac{1}{2}A = \sqrt{\frac{\sin s \sin(s - a)}{\sin b \sin c}}$$

$$\tan \frac{1}{2}A = \sqrt{\frac{\sin(s - b) \sin(s - c)}{\sin s \sin(s - a)}}$$

If $A + B + C = 2S$

$$\sin \frac{1}{2}a = \sqrt{\frac{-\cos S \cos(S - A)}{\sin B \sin C}}$$

$$\cos \frac{1}{2}a = \sqrt{\frac{\cos(S - B) \cos(S - C)}{\sin B \sin C}}$$

$$\tan \frac{1}{2}a = \sqrt{\frac{-\cos S \cos(S - A)}{\cos(S - B) \cos(S - C)}}$$

Delambre's Analogies

$$\sin \frac{1}{2}(A + B) \cos \frac{1}{2}c = \cos \frac{1}{2}(a - b) \cos \frac{1}{2}C$$

$$\sin \frac{1}{2}(A - B) \sin \frac{1}{2}c = \sin \frac{1}{2}(a - b) \cos \frac{1}{2}C$$

$$\cos \frac{1}{2}(A + B) \cos \frac{1}{2}c = \cos \frac{1}{2}(a + b) \sin \frac{1}{2}C$$

$$\cos \frac{1}{2}(A - B) \sin \frac{1}{2}c = \sin \frac{1}{2}(a + b) \sin \frac{1}{2}C$$

Section 3

COORDINATE GEOMETRY

3.1. DIVISION OF LINE IN GIVEN RATIO

P (x, y) divides the straight line AB in ratio $m:n$, i.e.

$$AP/PB = m/n$$

With A (x_1, y_1) and B (x_2, y_2),

$$x = \frac{nx_1 + mx_2}{m + n} \quad y = \frac{ny_1 + my_2}{m + n}$$

[For external division, the sign of m becomes negative.]

3.2. EQUATIONS OF A STRAIGHT LINE

Standard form $m = \tan \theta = \text{gradient or slope}$, $c = \text{intercept on } y \text{ axis}$

$$y = mx + c$$

Perpendicular form $p = \text{length of perpendicular from the origin O to the line}$ and α is the inclination of the perpendicular to Ox

$$x \cos \alpha + y \sin \alpha = p$$

Intercept form a, b are the intercepts on Ox, Oy respectively

$$\frac{x}{a} + \frac{y}{b} = 1$$

General form

$$Ax + By + C = 0$$

$$y - y_1 = m(x - x_1) \quad [\text{Line of slope } m \text{ passing through } (x_1, y_1)]$$

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1} \right) (x - x_1) \quad [\text{Line passing through } (x_1, y_1) \text{ and } (x_2, y_2)]$$

Angle between two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ is

$$\tan^{-1} \frac{m_1 - m_2}{1 + m_1 m_2}$$

Two lines are perpendicular if $m_1 m_2 = -1$.

Laws from Plotted Results. Form of law modified to give straight line graph.

Law suspected	Other form of law	Quantity to be plotted along horizontal	Quantity to be plotted along vertical	Slope gives:	Intercept gives:
$y = ax^2 + b$	—————	x^2	y	a	b
$xy = ax + by$	$y = by/x + a$	y/x	y	b	a
$y = ax^n$	$\log_{10} y = n \log x + \log_{10} a$	$\log_{10} x$	$\log_{10} y$	n	$\log_{10} a$
$y = ae^{kx}$	$\log_{10} y = 0.4343kx + \log_{10} a$	x	$\log_{10} y$	$0.4343k$	$\log_{10} a$

3.3. CIRCLE

(i) $x^2 + y^2 = a^2$ Circle centre origin, radius a

(ii) $(x - h)^2 + (y - k)^2 = a^2$ Circle centre (h, k) , radius a

(iii) $x^2 + y^2 + 2gx + 2fy + c = 0$ General equation of circle

Tangent at (x_1, y_1) on circle (i):

$$xx_1 + yy_1 = a^2$$

Tangent at (x_1, y_1) on circle (ii):

$$(x - h)(x_1 - h) + (y - k)(y_1 - k) = a^2$$

Tangent at (x_1, y_1) on circle (iii):

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

3.4. CONIC SECTIONS

A conic section is the locus of a point which moves in a plane such that the ratio e (called the *eccentricity*) of its distance from a fixed point (the *focus*) to its distance from a fixed straight line (the *directrix*) is constant.

The locus is an ellipse, a parabola or a hyperbola according as

$$e < = > 1$$

Polar equation of a conic referred to its focus where l = semi-latus rectum

$$\frac{l}{r} = 1 + e \cos \theta$$

(a) Ellipse

Ellipse, centre origin, semi-axes, $a, b, e < 1$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$b^2 = a^2 (1 - e^2) \quad l = b^2/a$$

Tangent to above ellipse at (x_1, y_1) on it

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

(b) Parabola

$y^2 = 4ax$ Parabola, origin at vertex, Ox axis of symmetry

$yy_1 = 2a(x + x_1)$ Tangent at (x_1, y_1) on the parabola

$y = mx + \frac{a}{m}$ Tangent of slope m for any value of m

Parabolic arc between two points at the same level. Span l , height s at the mid-point referred to origin at one end:

$$y = \frac{4s}{l^2}x(l - x)$$

Area of segment of parabola $= \frac{2}{3} sl$

(c) Hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ Hyperbola, centre origin, semi-axes a, b

$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ Tangent at (x_1, y_1) on the hyperbola

$\frac{x}{a} \pm \frac{y}{b} = 0$ Equations of the asymptotes

$$b^2 = a^2(e^2 - 1) \quad (e > 1)$$

$xy = a^2$ Rectangular hyperbola, asymptotes are the coordinate axes

$y_1x + x_1y = 2a^2$ Tangent at (x_1, y_1) on rectangular hyperbola

(d) Focal Distances

The ellipse and the hyperbola have two foci and two directrices whilst the parabola has only one focus and one directrix. A

focal distance is the distance of a point on the conic from a focus.

Ellipse: the two focal distances are $a \pm ex$. Sum = $2a$.

Hyperbola: the two focal distances are $ex \pm a$. Difference = $2a$.

3.5. GEOMETRY OF SOLIDS

(a) Point

Let (x, y, z) and (r, θ, ϕ) be the Cartesian and Polar coordinates of a point P,

$$x = r \sin \theta \cos \phi \quad y = r \sin \theta \sin \phi \quad z = r \cos \theta$$

where $r = OP = \sqrt{(x^2 + y^2 + z^2)}$

θ = co-latitude and $\cos \theta = z/\sqrt{(x^2 + y^2 + z^2)}$

ϕ = longitude and $\tan \phi = y/x$

(b) Straight Line

Distance between two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2)

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

The Direction Cosines of a straight line PQ are the cosines of the angles between PQ and Ox, Oy, Oz and are denoted respectively by l, m, n .

$$l = \frac{x_2 - x_1}{d} \quad m = \frac{y_2 - y_1}{d} \quad n = \frac{z_2 - z_1}{d}$$

$$l^2 + m^2 + n^2 = 1$$

The angle θ between two straight lines whose direction cosines are (l_1, m_1, n_1) and (l_2, m_2, n_2) is given by

$$\cos \theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

Equation of the straight line passing through the point (a, b, c) having direction cosines (l, m, n) is

$$\frac{x - a}{l} = \frac{y - b}{m} = \frac{z - c}{n}$$

$(x, y, z) = 0$ represents the equation of a surface

(c) **Plane**—linear in x, y, z

General equation of a plane

$$ax + by + cz + d = 0$$

Intercept form: plane cuts the axes at distances a, b, c from O

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Perpendicular form: perpendicular from O to the plane has a length l and direction cosines (l, m, n)

$$lx + my + nz = p$$

Length of perpendicular from (x_1, y_1, z_1) to the above plane

$$lx_1 + my_1 + nz_1 - p$$

(d) **Surfaces of the Second Degree**

Surfaces of revolution

$$x^2 + y^2 = a^2 \quad \text{Right circular cylinder, radius } a$$

$$x^2 + y^2 = a^2 z^2 \quad \text{Right circular cone, vertex origin, axis Oz}$$

$$x^2 + y^2 + z^2 = a^2 \quad \text{Sphere, centre O, radius } a$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{Ellipsoid, semi-axes } a, b, c$$

Section 4

NON-DECIMAL ARITHMETIC

4.1. ARITHMETIC BASES

Decimal Notation: numbers are expressed as multiples of powers of 10.

10 is called the *base*. The symbols used are 1, 2, . . . , 9.

293 means $2 \times 10^2 + 9 \times 10 + 3$

Any other positive integer may be used as the base of a number system, e.g. in octal arithmetic, 537 means $5 \times 8^2 + 3 \times 8 + 7$.

Decimal equivalent: 351

More generally, in arithmetic base r , the number represented by

$$a_n a_{n-1} \dots a_2 a_1 a_0$$

where each of the digits $a_0, a_1, a_2, \dots, a_{n-1}, a_n < r (a_n \neq 0)$, is

$$a_n r^n + a_{n-1} r^{n-1} + \dots + a_2 r^2 + a_1 r + a_0$$

Binary arithmetic (base 2), using the two digits 0, 1 only, has become very important because it is widely used in digital computers.

4.2. CHANGE OF BASE

N is an integral number expressed in any base. It is required to express N in base r .

$$\text{Let } N = a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0.$$

Divide N by r so that $N/r = a_n r^{n-1} + a_{n-1} r^{n-2} + \dots + a_1$; remainder a_0 .

Divide the quotient by r . The next remainder is a_1 and so on. The successive remainders are the required digits $a_0, a_1, \dots, a_n$. For example, by successive divisions by 2, decimal 247 becomes 11110111 in binary. Binary numbers are most easily converted to decimal by reversing the process

$$\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ & 3 & 7 & 15 & 30 & 61 & 123 & 247 \end{array}$$

Multiply the left-hand digit by 2, add the next binary digit, and continue this process.

4.3. FRACTIONS

In decimal arithmetic, 9.367 means $9 + \frac{3}{10} + \frac{6}{10^2} + \frac{7}{10^3}$.

Similarly in binary arithmetic, 1.101 means $1 + \frac{1}{2} + \frac{0}{2^2} + \frac{1}{2^3}$.

In arithmetic base r , the fraction $0.b_1 b_2 b_3 \dots$ where $b_1, b_2, b_3, \dots$ are integers, each less than r , of which one or more may be zero, is equivalent to

$$\frac{b_1}{r} + \frac{b_2}{r^2} + \frac{b_3}{r^3} + \dots$$

To express any fraction F in the scale of r , $b_1, b_2, b_3, \dots$ must be found such that

$$F = \frac{b_1}{r} + \frac{b_2}{r^2} + \frac{b_3}{r^3} + \dots$$

Multiplying by r $Fr = b_1 + \frac{b_2}{r} + \frac{b_3}{r^2} + \dots$

so that b_1 is the *integral* part of Fr , the fractional part F_1 being given by

$$F_1 = \frac{b_2}{r} + \frac{b_3}{r^2} + \dots \quad \therefore F_1 r = b_2 + \frac{b_3}{r} + \dots$$

and b_2 is the integral part of $F_1 r$ and so on.

For example, to express $11/16$ (decimal) as a binary fraction:

$$\frac{11}{16} \times 2 = \underline{1} + \frac{3}{8} \quad \frac{3}{8} \times 2 = \underline{0} + \frac{3}{4} \quad \frac{3}{4} \times 2 = \underline{1} + \frac{1}{2}$$

$$\frac{1}{2} \times 2 = \underline{1} + 0$$

$$11/16 \text{ (decimal)} = 0.1011 \text{ (binary)}$$

4.4. THEOREM

In the scale of r , the sum of the digits of any integer when divided by $(r - 1)$ gives the same remainder as when the integer itself is divided by $(r - 1)$.

Corollary: in the decimal scale ($r = 10$), a number is divisible by 9 if the sum of its digits is divisible by 9.

Section 5

COMPLEX NUMBERS

5.1. COMPLEX NUMBERS

If a and b are real numbers and $i = \sqrt{-1}$ (i.e. $i^2 = -1$), ib is known as an *imaginary* number and $z = a + ib$ is called a *complex* number and $\bar{z} = a - ib$ is called the *conjugate* of z .

If $z_1 = a + ib$ and $z_2 = c + id$, their sum, difference and product are

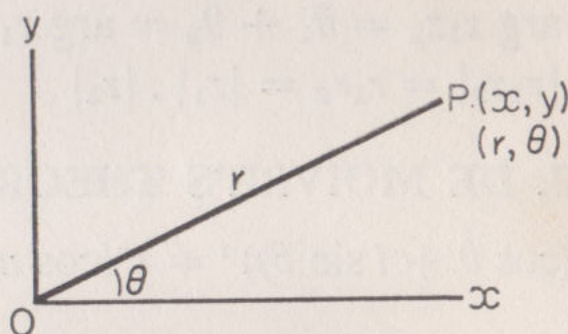
$$z_1 \pm z_2 = (a \pm c) + i(b \pm d).$$

$$z_1 z_2 = (ac - bd) + i(ad + bc)$$

$$z\bar{z} = a^2 + b^2$$

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} \\ &= \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2} \end{aligned}$$

If $z_1 = z_2$, then $a = c$, $b = d$.



Argand diagram

5.2. ARGAND DIAGRAM

There are two geometrical representations of the complex number $z = x + iy$:

(a) the point $P(x, y)$ (b) the vector $\overline{OP}$

$$z = x + iy = r(\cos \theta + i \sin \theta)$$

DEFINITIONS

(a) Absolute Value or Modulus of $z = |z| = |x + iy|$

$$= \text{length of } \overline{OP} = r = (x^2 + y^2)^{1/2}$$

(b) Argument or Amplitude of $z = \arg z$ or $\text{amp } z = \theta$

$$= \tan^{-1}(y/x) \text{ where } -\pi < \theta \leq \pi.$$

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \dots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)$$

$$= \cos \theta + i \sin \theta$$

Therefore $e^{-i\theta} = \cos \theta - i \sin \theta$ so that

$$\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \cosh i\theta$$

$$i \sin \theta = \frac{1}{2}(e^{i\theta} - e^{-i\theta}) = \sinh i\theta$$

Therefore $i \tan \theta = \tanh i\theta$

Also $z = re^{i\theta}$ and $\bar{z} = re^{-i\theta}$

$$z_1 z_2 = r_1(\cos \theta_1 + i \sin \theta_1) \cdot r_2(\cos \theta_2 + i \sin \theta_2)$$

$$= r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2}$$

$$= r_1 r_2 e^{i(\theta_1 + \theta_2)} = r_1 r_2 (\cos \overline{\theta_1 + \theta_2} + i \sin \overline{\theta_1 + \theta_2})$$

$$\arg z_1 z_2 = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$$

$$|z_1 z_2| = r_1 r_2 = |z_1| \cdot |z_2|$$

5.3. DE MOIVRE'S THEOREM

$$\{r(\cos \theta + i \sin \theta)\}^n = r^n(\cos n\theta + i \sin n\theta) \text{ for all values of } n$$

*n*th roots of z

$$z = x + iy = r(\cos \theta + i \sin \theta)$$

$$= r\{\cos(\theta + 2k\pi) + i \sin(\theta + 2k\pi)\} = re^{i(\theta + 2k\pi)}$$

where $k = 0, 1, 2, \dots, (n-1)$.

$$\sqrt[n]{z} = \sqrt[n]{(x + iy)} = r^{1/n} \left\{ \cos \left(\frac{\theta + 2k\pi}{n} \right) + i \sin \left(\frac{\theta + 2k\pi}{n} \right) \right\}$$

*n*th roots of unity

$$1 = \cos 0 + i \sin 0 \quad \text{therefore}$$

$$\sqrt[n]{1} = \cos \left(\frac{2k\pi}{n} \right) + i \sin \left(\frac{2k\pi}{n} \right) \quad [k = 0, 1, 2, \dots, (n-1)]$$

Cube roots of unity ($n = 3$)

$$\sqrt[3]{1} = \cos \left(\frac{2k\pi}{3} \right) + i \sin \left(\frac{2k\pi}{3} \right) \quad (k = 0, 1, 2)$$

The three roots are $1, -\frac{1}{2} \pm i\sqrt{3}/2$.

Section 6

MODERN ALGEBRA

6.1. SET

A set is a well-defined collection of objects. "Well-defined" means that it may readily be decided whether an object is a member of the set or not, e.g.

- (i) The set of all counties in the U.K. The *elements* of this set may be listed: {Aberdeenshire, Anglesey, ... Yorkshire}.
- (ii) The set of all positive even integers less than 16: {2, 4, 6, . . ., 14}.

A set may contain an infinite number of elements.

6.2. NOTATION

$x \in A$ means x is an element of the set A .

($x \notin A$ is the negative statement.)

$A = \{x_1, x_2, . . ., x_n\}$ means $x_1, x_2, . . ., x_n$ are all the elements of set A .

$A = \{x, P\}$ means A is set of elements x satisfying a property P .

$\exists$ means "there exist(s)"

For two statements p and q ,

$p \Rightarrow q$ means " p implies q " or "if p then q ".

$p \Leftrightarrow q$ means p and q are equivalent statements or " p if and only if q ".

$A = B$ means two sets A and B are equal, i.e. A and B contain precisely the same elements or $x \in A \Leftrightarrow x \in B$.

$A \sim B$ means sets A and B are *equivalent*, i.e. there is *one-to-one correspondence* between their elements.

$A \subset B$ means A is a *sub-set* of B , i.e. every element of A is also an element of B .

$$A \subset B \Leftrightarrow (x \in A \Rightarrow x \in B)$$

$$A = B \Leftrightarrow A \subset B \text{ and } B \subset A$$

6.3. OPERATIONS ON SETS

$A \cup B$ denotes the *union* of two sets A and B . It is the set of those elements which belong to either A or B or both A and B .

$$\text{i.e. } x \in A \text{ or } x \in B \Leftrightarrow x \in A \cup B$$

$A \cap B$ denotes the *intersection* of two sets A and B . It is the set of those elements which are members of both A and B .

$$\text{i.e. } x \in A \text{ and } x \in B \Leftrightarrow x \in A \cap B$$

The *universal set* V is the set of all possible objects in a given problem. The *empty* or *null set* ϕ is the set having no elements.

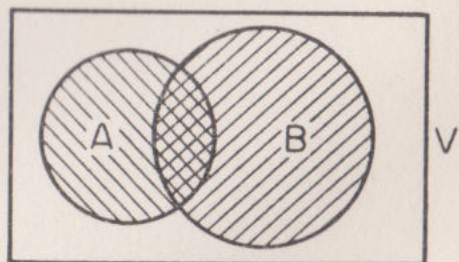
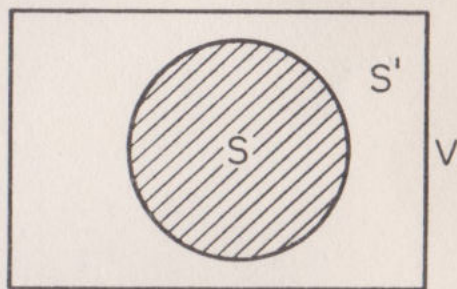
If two sets A and B have no elements in common, $A \cap B = \phi$.

6.4. VENN DIAGRAMS

Let the universal set V be represented by a rectangle and let any subset S of V be represented by a circle (shaded) drawn inside the rectangle. The set S' (unshaded) of all elements not in S is called the complement of S and $S \cup S' = V$.

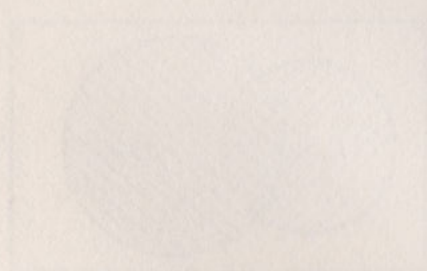
$A \cap B$ is represented by the doubly cross-hatched area.

$A \cup B$ is represented by the whole shaded area



The following important results can easily be verified by Venn diagrams:

- (i) $\left. \begin{aligned} (A \cap B) \cap C &= A \cap (B \cap C) \\ (A \cup B) \cup C &= A \cup (B \cup C) \end{aligned} \right\} \text{Associative Laws}$
- (ii) $\left. \begin{aligned} A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \\ A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \end{aligned} \right\} \text{Distributive Laws}$
- (iii) $\left. \begin{aligned} (A \cup B)' &= A' \cap B' \\ (A \cap B)' &= A' \cup B' \end{aligned} \right\} \text{De Morgan's Laws}$



Section 7

MATRICES

7.1. THE ORDER OF A MATRIX

A matrix is an *ordered* array of numbers or other elements. A matrix having m rows and n columns is said to be of order $m \times n$.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$$

is a 4×3 matrix.

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \text{ is a column matrix; order } 3 \times 1.$$

$(-2 \quad 1 \quad 0 \quad 7)$ is a row matrix; order 1×4 .

A matrix having n rows and n columns is called a *square* matrix of order n . Matrices will be denoted by heavy capital letters $A, B, C, \dots$

7.2. THE MULTIPLICATION OF MATRICES

Suppose A is a 3×5 matrix and B a 5×4 matrix. Then, for example, the element c_{23} of the product $A \times B$ is obtained by the multiplication of the elements of the *second* row of A by the corresponding elements of the *third* column of B , adding the products so that

$$\begin{aligned} c_{23} &= a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} + a_{24}b_{43} + a_{25}b_{53} \\ &= \sum_{k=1}^5 a_{2k}b_{k3} \end{aligned}$$

In general

$$c_{ij} = \sum_{k=1}^5 a_{ik}b_{kj}$$

Compatibility

The above product rule is applicable only if the number of *columns* of **A** equals the number of *rows* of **B**. Matrices which satisfy this criterion are said to be *compatible* for multiplication; otherwise, their product is not defined.

In general, a matrix **A** ($m \times n$) is compatible with a matrix **B** ($n \times p$) and $\mathbf{A} \times \mathbf{B}$ is a matrix **C** ($m \times p$).

A matrix **A** ($m \times n$) is compatible with a matrix **B** ($n \times m$) for multiplication in either order but $\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$ in general.

7.3. UNIT MATRIX: NULL OR ZERO MATRIX

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

If any (2×2) matrix is pre- or post-multiplied by $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, it is unchanged.

I is called the *unit* matrix of order 2.

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a unit matrix of order 3, and so on to higher orders.

A zero matrix has all its elements zero.

7.4. EQUALITY; ADDITION; MULTIPLICATION BY A SCALAR

(a) Two matrices are equal if and only if they are of the same order and their corresponding elements are equal.

(b) The sum of two matrices of the same order is defined to be a matrix whose elements are the sums of the corresponding elements of these two matrices.

(c) The product of a matrix by a number (scalar) k is defined to be the matrix derived from the first by multiplying each of its elements by k , e.g.

$$\text{If } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ then } kA = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}.$$

7.5. DETERMINANT OF A SQUARE MATRIX

Matrix A ($n \times n$)

Determinant of matrix $A = |A|$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{vmatrix}$$

A rule assigns a value to the determinant $|A|$ of order n .

Let α_{ij} be the square matrix of order $(n - 1)$ obtained by deleting the i th row and the j th column of A . The determinant $|\alpha_{ij}|$ is called the *minor* of a_{ij} in $|A|$. The *cofactor* A_{ij} of a_{ij} is defined by

$$A_{ij} = (-1)^{i+j} \times \text{value of } |\alpha_{ij}|$$

The value assigned to the determinant $|A|$ is

$$|A| = \sum_{j=1}^n a_{ij} A_{ij}$$

(This formula will give the same value of $|A|$ irrespective of the particular row in terms of which it is evaluated.)

If the products obtained by multiplying the elements of any

row by the corresponding cofactors of another row are added, the result is zero.

$$\sum_{k=1}^n a_{ik} A_{jk} = 0 \quad (i \neq j)$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

(2 × 2)

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} = a_{11}a_{22} + a_{12}(-a_{21})$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

(3 × 3)

$$\begin{aligned} |\mathbf{A}| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \end{aligned}$$

A matrix whose determinant is zero is a *singular* matrix.

One of the rules for multiplying two determinants of the same order is precisely the rule (Sec. 7.2) for multiplying two square matrices of the same order. Hence if $\mathbf{A}$, $\mathbf{B}$ are square matrices of the same order

$$|\mathbf{AB}| = |\mathbf{A}| \cdot |\mathbf{B}| \quad \text{and} \quad |\mathbf{AB}| = |\mathbf{BA}|$$

even though, in general, $\mathbf{AB} \neq \mathbf{BA}$.

7.6. INVERSE MATRIX

The inverse of a square matrix $\mathbf{A}$ is $\mathbf{A}^{-1}$ of the same order such that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{AA}^{-1} = \mathbf{I}$$

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

A singular matrix has no inverse.

7.7. MATRIX SOLUTION OF LINEAR EQUATIONS

Equations $ax + by = p$ and $cx + dy = q$

Matrix form: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$

Solution: $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{|A|} \begin{pmatrix} pd - qb \\ aq - cp \end{pmatrix}$

7.8. THE TRANSPOSE OF A MATRIX

If the rows and columns of matrix A ($m \times n$) are interchanged, the resulting matrix A' ($n \times m$) is said to be the transpose of A .

A square matrix A is *symmetric* if $A = A'$ and *skew symmetric* if $A' = -A$.

If $A = A'$ (symmetric), the elements a_{ij} of A are such that $a_{ji} = a_{ij}$.

If $A' = -A$ (skew-symmetric) $a_{ij} = -a_{ji}$ ($i \neq j$), whilst $a_{ii} = 0$.

7.9. ADJUGATE OR ADJOINT MATRIX

If A is a square matrix, the *adjugate matrix* $\text{adj } A$ is formed by replacing each element of A by its cofactor in $|A|$ and transposing, so that

$$\text{adj } A = (A_{ij})' = (A_{ji})$$

where (A_{ij}) denotes a matrix of order n having elements A_{ij} .

$$\text{If } A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix} \quad \text{then} \quad \text{adj } A = \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ \vdots & & & \\ \vdots & & & \\ A_{1n} & A_{2n} & & A_{nn} \end{pmatrix}$$

7.10. INVERSE OF A SQUARE MATRIX

If A is a non-singular matrix of order n ,

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

7.11. SOLUTION OF LINEAR EQUATIONS

(a) $a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i \quad (i = 1, 2, \dots, n)$

are n simultaneous linear equations in n unknowns x_i .

Matrix form $AX = B$

where A is a square matrix of the coefficients a_{ij} and X, B are column matrices of the x_i and b_i .

If A is non-singular, $X = A^{-1}B$ where A^{-1} is given in 7.10.

(b) Method of Triangular Decomposition illustrated for three unknowns.

Matrix form $AX = B$ (A is the 3×3 matrix of a_{ij})

Find $V = \begin{pmatrix} 1 & 0 & 0 \\ v_{21} & 1 & 0 \\ v_{31} & v_{32} & 1 \end{pmatrix}$ and $U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$

such that $VU = A$.

From the equations, $u_{11} = a_{11}, u_{12} = a_{12}, \dots, v_{21}u_{11} = a_{21}, \dots$, etc.

Let $VY = B$ (Y is a 3×1 matrix) whence $y_1 = b_1$ and

$$v_{21}y_1 + y_2 = b_2 \quad v_{31}y_1 + v_{32}y_2 + y_3 = b_3$$

from which Y is found.

$$AX = B \quad VUX = VY \quad UX = Y$$

whence $u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = y_1$ and

$$u_{22}x_2 + u_{23}x_3 = y_2 \quad u_{33}x_3 = y_3$$

from which X is found.

Section 8

VECTORS

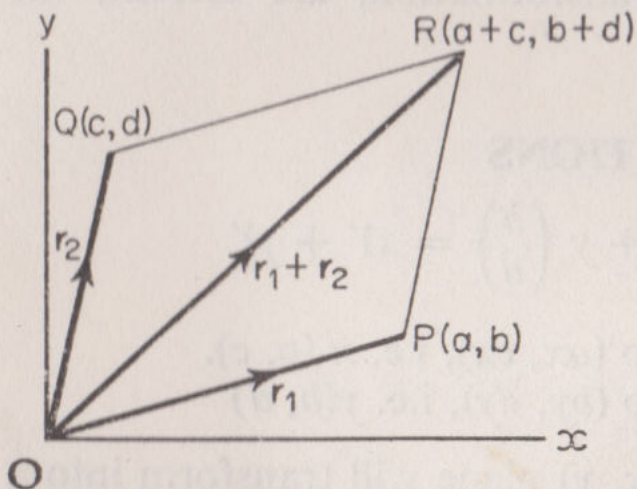
8.1. COLUMN VECTOR

Position vector of $P(a, b) = \overrightarrow{OP} = \mathbf{r}_1$.

$\mathbf{r}_1$ may be represented by a column matrix or *column vector* $\begin{pmatrix} a \\ b \end{pmatrix}$.

Addition of Two Vectors. Let $\mathbf{r}_2 = \begin{pmatrix} c \\ d \end{pmatrix}$, then

$$\mathbf{r}_1 + \mathbf{r}_2 = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a + c \\ b + d \end{pmatrix} \text{ (Parallelogram Law)}$$



Base Vectors

$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represent unit vectors along Ox , Oy respectively.

$$\mathbf{r} = \begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a\mathbf{i} + b\mathbf{j}$$

8.2. LINEAR TRANSFORMATIONS IN A PLANE

A column vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is transformed into another column vector $\begin{pmatrix} x' \\ y' \end{pmatrix}$ on pre-multiplying the former by a 2×2 matrix A (transformation matrix).

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

i.e. $\mathbf{r}' = A\mathbf{r}$ and $x' = ax + by$, $y' = cx + dy$

The base vectors $\mathbf{i}, \mathbf{j}$ transform into $\mathbf{i}', \mathbf{j}'$ such that

$$A\mathbf{i} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = \mathbf{i}' \quad \text{and} \quad A\mathbf{j} = \begin{pmatrix} b \\ d \end{pmatrix} = \mathbf{j}'$$

i.e. $\mathbf{i}', \mathbf{j}'$ are the columns of A . Thus if the images of the points $(1, 0)$ and $(0, 1)$, under a transformation, are known, the transformation is known.

8.3. AFFINE TRANSFORMATIONS

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = x \begin{pmatrix} a \\ c \end{pmatrix} + y \begin{pmatrix} b \\ d \end{pmatrix} = x\mathbf{i}' + y\mathbf{j}'$$

The point $(x, 0)$ transforms into (ax, cx) , i.e. $x(a, c)$.

The point $(0, y)$ transforms into (by, dy) , i.e. $y(b, d)$

A grid of unit squares in the (x, y) plane will transform into a grid of parallelograms. The transformation is such that parallel lines remain parallel and the *ratios* of lengths in the same direction are unchanged. It is called an *affine* transformation.

Exceptionally, if the matrix A is *singular*, i.e. if $|A| = 0$, then $a/c = b/d$ or $a = kb$, $c = kd$ so that $x' = (b/d)y'$ and all points in the (x, y) plane are transformed into points on one straight line.

8.4. BASIC TRANSFORMATIONS

<i>Type</i>	<i>Matrix</i>	<i>Algebraic Equivalent</i>
Reflection in Ox	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$x' = x$ $y' = -y$
Reflection on Oy	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$x' = -x$ $y' = y$
Reflection in line $y = x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$x' = y$ $y' = x$
Reflection in line $y = -x$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$x' = -y$ $y' = -x$
Quarter turn about origin	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$x' = -y$ $y' = x$
Half turn about origin	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$x' = -x$ $y' = -y$
Three-quarters turn about origin	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$x' = y$ $y' = -x$
Enlargement	$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$	$x' = kx$ $y' = ky$
Shearing parallel to Ox	$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$	$x' = x + ky$ $y' = y$
Rotation through angle α	$\begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$	$x' = x \cos \alpha - y \sin \alpha$ $y' = x \sin \alpha + y \cos \alpha$
Reflection in line $y = x \tan \alpha$	$\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$	$x' = x \cos 2\alpha + y \sin 2\alpha$ $y' = x \sin 2\alpha - y \cos 2\alpha$

8.5. EIGENVECTORS

Consider the transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where the transformation matrix A is non-singular. Vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ whose directions are unchanged by the transformation are

called *eigenvectors*, i.e. $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} kx \\ ky \end{pmatrix}$

We require solutions of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = k \begin{pmatrix} x \\ y \end{pmatrix}$

$$\text{i.e. of } (A - kI) \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

The *eigenvalues* of A are the values of k obtained from the determinantal equation

$$|A - kI| = 0$$

Section 9

DIFFERENTIAL CALCULUS

9.1. LIMITS

A function $f(x)$ is said to approach the limit L as x approaches c , i.e. $\lim_{x \rightarrow c} f(x) = L$ if, given any positive number ε , there is a positive number δ such that

$$|f(x) - L| < \varepsilon \quad \text{when } 0 < |x - c| < \delta$$

Some important limits

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1$$

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

9.2. DIFFERENTIATION

Let $y = f(x)$ and let δy , δx be corresponding increments in y , x then the differential coefficient or derivative of y with respect to x is given by

$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

= gradient of the curve $y = f(x)$ at the point (x, y) on it.

Other symbols for this differential coefficient are y' , Dy , and $f'(x)$.

Rules for Differentiation

Let u, v, w , etc. be functions of x .

$$y = cu \quad (c = \text{const}) \quad \frac{dy}{dx} = c \frac{du}{dx}$$

$$y = u + v + w + \dots \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \dots$$

$$y = uv \quad \frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$y = \frac{u}{v} \quad \frac{dy}{dx} = \frac{1}{v^2} \left(v \frac{du}{dx} - u \frac{dv}{dx} \right)$$

$$y = uvw \quad \frac{dy}{dx} = uvw \left(\frac{1}{u} \frac{du}{dx} + \frac{1}{v} \frac{dv}{dx} + \frac{1}{w} \frac{dw}{dx} \right)$$

$$y = u^n \quad \frac{dy}{dx} = nu^{n-1} \frac{du}{dx}$$

$$y = f(u) \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

where $u = \phi(x)$

If y and x are both functions of t , $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

If x, y are distances and t is time, $\frac{dx}{dt}, \frac{dy}{dt}$ (often denoted by $\dot{x}, \dot{y}$) are velocities.

Table of Important Differential Coefficients

y	dy/dx
x^n	nx^{n-1}
e^{kx}	ke^{kx}
$\sin(ax + b)$	$a \cos(ax + b)$
$\cos(ax + b)$	$-a \sin(ax + b)$
$\log_e(ax + b)$	$a/(ax + b)$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sin^{-1} x/a$	$1/\sqrt{a^2 - x^2}$
$\cos^{-1} x/a$	$-1/\sqrt{a^2 - x^2}$
$\tan^{-1} x/a$	$a/(x^2 + a^2)$
$\sinh x = \frac{1}{2}(e^x - e^{-x})$	$\cosh x = \frac{1}{2}(e^x + e^{-x})$
$\cosh x$	$\sinh x$
$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$	$\operatorname{sech}^2 x = 1/\cosh^2 x$

9.3. NEWTON'S METHOD OF APPROXIMATION

If x_0 is an approximation to a root of $f(x) = 0$, a better approximation x_1 is given by

$$\frac{f(x_0)}{x_0 - x_1} = f'(x_0)$$

so that $x_1 = x_0 - f(x_0)/f'(x_0)$.

9.4. SUCCESSIVE DIFFERENTIATION

If dy/dx is differentiated with respect to x , the second derivative d^2y/dx^2 is obtained. Successive differentiations produce derivatives of higher orders.

Notation

$$\text{If } y = f(x), \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2y}{dx^2} = \frac{d}{dx} [f'(x)] = f''(x)$$

$$\frac{d}{dx} \left(\frac{d^2y}{dx^2} \right) = \frac{d^3y}{dx^3} = f'''(x)$$

$$n\text{th derivative} = \frac{d^ny}{dx^n} = D^ny = f^{(n)}(x)$$

Leibniz' Formula for the n th derivative of a product.

$$\begin{aligned} D^n(uv) &= D^nu \cdot v + \binom{n}{1} D^{n-1}u \cdot Dv + \binom{n}{2} D^{n-2}u \cdot D^2v + \dots \\ &+ \binom{n}{r} D^{n-r}u \cdot D^rv + \dots + \binom{n}{n-1} Du \cdot D^{n-1}v + u \cdot D^nv \end{aligned}$$

9.5. MAXIMUM AND MINIMUM VALUES

If $dy/dx = 0$ at $x = a$, then

$y = f(x)$ has a minimum at $x = a$ if $d^2y/dx^2 > 0$ at $x = a$

$y = f(x)$ has a maximum at $x = a$ if $d^2y/dx^2 < 0$ at $x = a$

If $d^2y/dx^2 = 0$ at $x = a$, the graph of $y = f(x)$ has a *point of inflection* at $x = a$.

9.6. CURVATURE

$y = f(x)$ graph APB

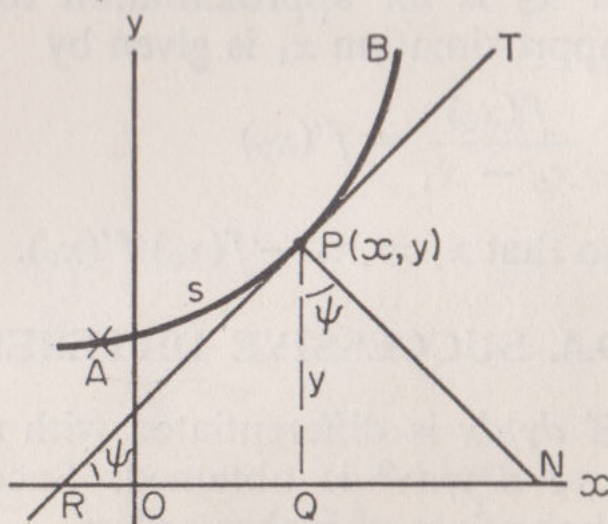
PT, PN are tangent and normal at P respectively.

$$dy/dx = \tan \psi$$

= slope of tangent

$$\begin{aligned} \text{Subtangent QR} &= \frac{y}{dy/dx} \\ &= y/y' \end{aligned}$$

$$\text{Subnormal QN} = y \frac{dy}{dx} = yy'$$



Length of tangent = PR = $y(1 + y'^2)^{1/2}/y'$

Length of normal = PN = $y(1 + y'^2)^{1/2}$

Let s be the length of the arc AP measured from a fixed point A.

$$\text{Curvature} = \kappa = \frac{d\psi}{ds} = y'' (1 + y'^2)^{-3/2}$$

$$\text{Radius of curvature} = 1/\kappa$$

If the curve is nearly horizontal at all points as in the case of horizontal beams, dy/dx is small compared with unity and the curvature is d^2y/dx^2 approximately.

9.7. PARTIAL DIFFERENTIATION

$$\text{If } z = f(x, y), dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

dx, dy, dz are called *differentials*.

$$\text{If } x, y, z \text{ are all functions of } t, \frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

If $u = f(x_1, x_2, x_3, \dots, x_n)$, then

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \frac{\partial u}{\partial x_3} dx_3 + \dots + \frac{\partial u}{\partial x_n} dx_n$$

Implicit Functions If y is an implicit function of x as given by $f(x, y) = 0$,

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0 \quad \therefore \frac{dy}{dx} = - \frac{\partial f / \partial x}{\partial f / \partial y}$$

9.8. INFINITE SERIES

A series may contain either a finite or an infinite number of terms. Let S_n denote the sum of n terms of a series. The series is said to be *convergent* or *divergent* according as the $\lim_{n \rightarrow \infty} S_n$ is a finite quantity or is $\pm \infty$.

Binomial Series With reference to 1.10, if n is a positive integer,

$$\begin{aligned}(1+x)^n &= \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{r}x^r \\ &\quad + \dots + \binom{n}{n}x^n \\ &= 1 + nx + \frac{1}{2!}n(n-1)x^2 + \dots + \\ &\quad \frac{1}{r!}n(n-1)\dots(n-r+1)x^r + \dots + x^n\end{aligned}$$

which is a finite series.

If n is *not* a positive integer

$$\begin{aligned}(1+x)^n &= 1 + nx + \frac{1}{2!}n(n-1)x^2 + \dots + \\ &\quad \frac{1}{r!}n(n-1)\dots(n-r+1)x^r + \dots \text{ad inf.}\end{aligned}$$

provided $|x| < 1$, i.e. the *infinite* series converges to $(1+x)^n$
provided $|x| < 1$, otherwise it is *divergent*.

Approximations. If x, y, z, u, v are small compared with 1,

$$(1+x)^n \simeq 1 + nx \quad \text{for all } n$$

$$\text{e.g. } \sqrt{1+x} \simeq 1 + \frac{1}{2}x$$

$$\frac{(1+x)(1+y)(1+z)}{(1+u)(1+v)} \simeq 1 + x + y + z - u - v$$

Exponential Series

$$e^{kx} = 1 + (kx) + \frac{1}{2!}(kx)^2 + \frac{1}{3!}(kx)^3 + \frac{1}{4!}(kx)^4 + \dots \text{ad inf.}$$

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots = 2.71828 \dots$$

Trigonometrical Series

$$\sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \dots \text{ad inf.}$$

$$\cos x = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \dots \text{ad inf.}$$

$$\tan x = x + \frac{1}{3} x^3 + \frac{2}{15} x^5 + \frac{17}{315} x^7 + \dots \text{ad inf.}$$

Maclaurin's Theorem

$$f(x) = f(0) + xf'(0) + \frac{1}{2!} x^2 f''(0) + \dots + \frac{1}{n!} x^n f^{(n)}(0) + \dots$$

Taylor's Theorem

Single Independent Variable

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2!} h^2 f''(x) + \dots + \frac{1}{n!} h^n f^{(n)}(x) + \dots$$

Two Independent Variables

$$\begin{aligned} f(x+h, y+k) = f(x, y) &+ \left(h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y} \right) \\ &+ \frac{1}{2!} \left(h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2} \right) + \dots \end{aligned}$$

9.9. MAXIMUM AND MINIMUM VALUES

By Taylor's Theorem, necessary conditions for a maximum or a minimum of $f(x, y)$ are $\partial f / \partial x = \partial f / \partial y = 0$.

Provided that $\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} > \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2$, then $f(x, y)$ is a maximum

or minimum according as $\partial^2 f / \partial x^2$ and $\partial^2 f / \partial y^2$ are both negative or both positive.

Section 10

INTEGRAL CALCULUS

10.1. STANDARD INTEGRALS

$f(x)$	$\int f(x)dx$
x^n	$x^{n+1}/(n+1)$ except when $n = -1$
$1/x$	$\log_e x$
$1/(ax+b)$	$\frac{1}{a} \log_e (ax+b)$
e^{kx}	$\frac{1}{k} e^{kx}$
a^{kx}	$a^{kx}/k \log_e a$
$\sin(ax+b)$	$-\frac{1}{a} \cos(ax+b)$
$\cos(ax+b)$	$\frac{1}{a} \sin(ax+b)$
$\tan(ax+b)$	$\frac{1}{a} \log \sec(ax+b)$
$\sec^2 x$	$\tan x$
$\operatorname{cosec}^2 x$	$-\cot x$
$\sec x \tan x$	$\sec x$
$\operatorname{cosec} x \cot x$	$-\operatorname{cosec} x$

$f(x)$	$\int f(x)dx$
$1/\sqrt{a^2 - x^2}$	$\sin^{-1}(x/a)$
$1/\sqrt{x^2 + a^2}$	$\log \frac{1}{a} [x + \sqrt{x^2 + a^2}] = \sinh^{-1}(x/a)$
$1/\sqrt{x^2 - a^2}$	$\log \frac{1}{a} [x + \sqrt{x^2 - a^2}] = \cosh^{-1}(x/a)$
$1/(x^2 + a^2)$	$\frac{1}{a} \tan^{-1}(x/a)$
$1/(x^2 - a^2)$	$\frac{1}{2a} \log \left(\frac{x-a}{x+a} \right) \quad (x > a)$
$1/(a^2 - x^2)$	$\frac{1}{2a} \log \left(\frac{a+x}{a-x} \right) \quad (x < a)$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\operatorname{sech}^2 x$	$\tanh x$
$f'(x)/f(x)$	$\log f(x)$
$f'(x)/\sqrt{f(x)}$	$2\sqrt{f(x)}$

Integration by Parts

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$$\text{Hence } \int \log(x) dx = x \log x - x$$

10.2. REDUCTION FORMULAE

$$\int \cos^n \theta d\theta = \frac{1}{n} \sin \theta \cos^{n-1} \theta + \frac{n-1}{n} \int \cos^{n-2} \theta d\theta$$

$$\int \sin^n \theta d\theta = -\frac{1}{n} \cos \theta \sin^{n-1} \theta + \frac{n-1}{n} \int \sin^{n-2} \theta d\theta$$

$$\begin{aligned}
& \int \sin^m \theta \cos^n \theta d\theta \\
&= -\frac{1}{m+n} \sin^{m-1} \theta \cos^{n+1} \theta + \frac{m-1}{m+n} \int \sin^{m-2} \theta \cos^n \theta d\theta \\
&= \frac{1}{m+n} \sin^{m+1} \theta \cos^{n-1} \theta + \frac{n-1}{m+n} \int \sin^m \theta \cos^{n-2} \theta d\theta \\
&\int \tan^n \theta d\theta = \frac{1}{n-1} \tan^{n-1} \theta - \int \tan^{n-2} \theta d\theta \\
&\int \cot^n \theta d\theta = -\frac{1}{n-1} \cot^{n-1} \theta - \int \cot^{n-2} \theta d\theta
\end{aligned}$$

10.3. DEFINITE INTEGRALS

$$\int_0^{2\pi n/p} \sin pt dt = \int_0^{2\pi n/p} \cos pt dt = 0 \quad n \text{ integer}$$

$$\int_0^{\pi/2p} \sin pt dt = \int_0^{\pi/2p} \cos pt dt = 1/p$$

$$\int_0^{\pi/2p} \sin^2 pt dt = \int_0^{\pi/2p} \cos^2 pt dt = \pi/4p$$

$$\int_0^{\pi/2} \cos^n \theta d\theta = \frac{n-1}{n} \int_0^{\pi/2} \cos^{n-2} \theta d\theta$$

$$\int_0^{\pi/2} \sin^n \theta d\theta = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} \theta d\theta$$

$$\begin{aligned}
\int_0^{\pi/2} \sin^m \theta \cos^n \theta d\theta &= \frac{m-1}{m+n} \int_0^{\pi/2} \sin^{m-2} \theta \cos^n \theta d\theta \\
&= \frac{n-1}{m+n} \int_0^{\pi/2} \sin^m \theta \cos^{n-2} \theta d\theta
\end{aligned}$$

If p and q are integers, $p \neq q$,

$$\begin{aligned}
\int_0^{2\pi} \sin pt \sin qt dt &= \int_0^{2\pi} \sin pt \cos qt dt \\
&= \int_0^{2\pi} \cos pt \cos qt dt = 0
\end{aligned}$$

If $p = q$ these become respectively.

$$\int_0^{2\pi} \sin^2 pt \, dt = \pi \quad \int_0^{2\pi} \frac{1}{2} \sin 2pt \, dt = 0$$

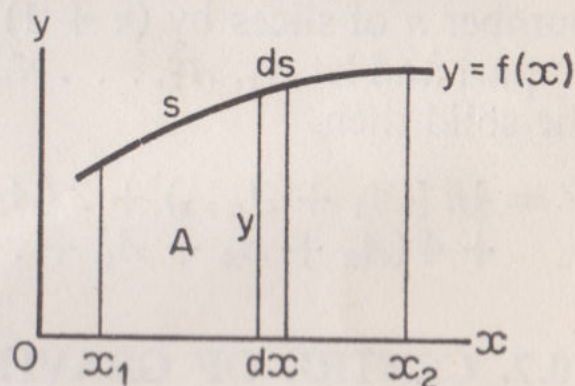
$$\int_0^{2\pi} \cos^2 pt \, dt = \pi$$

10.4. AREA BOUNDED BY A CURVE; LENGTH OF CURVE

Cartesian Coordinates

$$\text{Area } A = \int_{x_1}^{x_2} f(x) dx$$

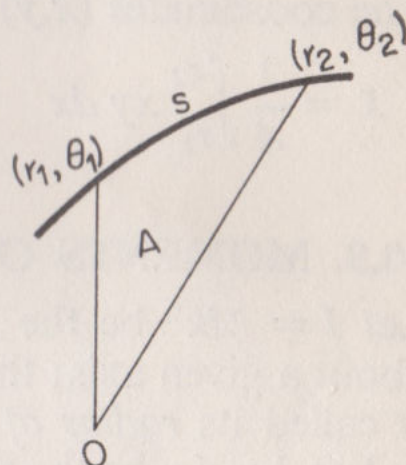
$$\text{Arc } s = \int_{x_1}^{x_2} \sqrt{1 + y'^2} dx$$



Polar Coordinates

$$\text{Area } A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 d\theta$$

$$\begin{aligned} \text{Arc } s &= \int_{\theta_1}^{\theta_2} \sqrt{r^2 + (dr/d\theta)^2} d\theta \\ &= \int_{r_1}^{r_2} \sqrt{1 + r^2 (d\theta/dr)^2} dr \end{aligned}$$



10.5. VOLUMES AND SURFACES OF REVOLUTION

When revolved about Ox , the curve $y = f(x)$ generates a volume and a surface of revolution.

$$\text{Volume} = \int_{x_1}^{x_2} \pi y^2 dx$$

$$\text{Area} = \int 2\pi y ds = \int_{x_1}^{x_2} 2\pi y \sqrt{1 + y'^2} dx$$

10.6. SIMPSON'S RULE FOR AREA AND VOLUME

Area Under a Curve Let the area be divided up into an *even* number n of equally wide strips (width h) by ordinates $y_1, y_2, \dots, y_{n+1}$ then

$$A = \frac{1}{3}h [(y_1 + y_{n+1}) + 2(y_3 + y_5 + y_7 + \dots) + 4(y_2 + y_4 + y_6 + \dots)]$$

Volume of a Solid Let the solid be divided up into an *even* number n of slices by $(n + 1)$ parallel planes at equal distances h apart and let $A_1, A_2, \dots, A_{n+1}$ be the areas of the sections of the solid then

$$V = \frac{1}{3}h [(A_1 + A_{n+1}) + 2(A_3 + A_5 + A_7 + \dots) + 4(A_2 + A_4 + A_6 + \dots)]$$

10.7. CENTRE OF GRAVITY

The coordinates $(\bar{x}, \bar{y})$ of the centre of gravity of an area are

$$\bar{x} = \frac{1}{A} \int_{x_1}^{x_2} xy \, dx \quad \bar{y} = \frac{1}{A} \int_{x_1}^{x_2} \frac{1}{2} y^2 \, dx$$

10.8. MOMENTS OF INERTIA

Let $I = Mk^2$ be the moment of inertia of a body of mass M about a given axis; then k , which has the dimension of a length, is called its *radius of gyration* about that axis.

Let I_{Ox}, I_{Oy} be the moments of inertia of the plane area A of uniform density σ about Ox, Oy , then

$$I_{Ox} = \int_{x_1}^{x_2} \sigma \frac{1}{3} y^3 \, dx \quad I_{Oy} = \int_{x_1}^{x_2} \sigma x^2 y \, dx$$

Theorem of Perpendicular Axes (applicable to plane laminae)

Let Ox, Oy be a pair of perpendicular axes lying in the *plane* of an area A and let Oz be an axis perpendicular to both, then

$$I_{Oz} = I_{Ox} + I_{Oy}$$

Theorem of Parallel Axes Let OO' , GG' be a pair of parallel axes at a distance h apart and let GG' pass through the centre of gravity of a body of mass M , then

$$I_{OO'} = I_{GG'} + Mh^2$$

Let the solid of revolution and the surface of revolution have uniform densities ρ and σ respectively then

$$\text{Solid } I_{Ox} = \rho \int_{x_1}^{x_2} \frac{1}{2} \pi y^4 dx \quad I_{Oy} = \rho \int_{x_1}^{x_2} \pi y^2 (\frac{1}{2} y^2 + x^2) dx$$

$$\text{Surface } I_{Ox} = \sigma \int 2\pi y^3 ds = \sigma \int_{x_1}^{x_2} 2\pi y^3 \sqrt{1 + y'^2} dx$$

$$I_{Oy} = \sigma \int_{x_1}^{x_2} 2\pi y (\frac{1}{2} y^2 + x^2) \sqrt{1 + y'^2} dx$$

10.9. THEOREMS OF PAPPUS AND GULDIN

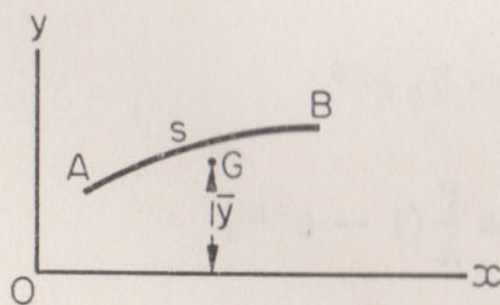
(i) Arc AB of length s does not intersect Ox .

G is centre of gravity of the arc.

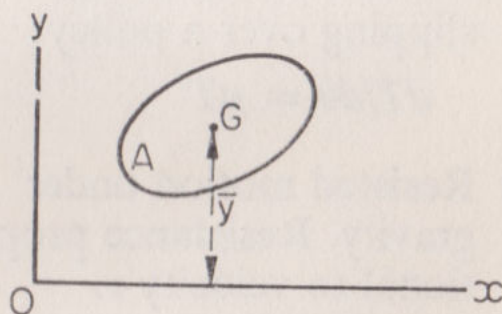
Surface generated by the revolution of AB about Ox is $s \times 2\pi \bar{y}$.

(ii) Area A does not intersect Ox . G is the centre of gravity of A .

Volume generated by the revolution of A about Ox is $A \times 2\pi \bar{y}$.



(i)



(ii)

Section 11

DIFFERENTIAL EQUATIONS

11.1. FIRST ORDER EQUATIONS

Equation

Solution

(a) Variables Separable

$$f(x) \frac{dy}{dx} = \phi(y)$$

$$\int \frac{dy}{\phi(y)} = \int \frac{dx}{f(x)} + C$$

$$f(y) \frac{dy}{dx} = \phi(x)$$

$$\int f(y) dy = \int \phi(x) dx + C$$

Examples

(i) Decay Function:

$$dx/dt = -kx$$

$$x = Ce^{-kt}$$

(ii) Tension in a belt just slipping over a pulley:

$$dT/d\theta = \mu T$$

$$T = T_0 e^{\mu\theta}$$

(iii) Resisted motion under gravity. Resistance proportional to velocity v .

$$dv/dt = g - kv$$

$$v = \frac{g}{k} (1 - e^{-kt})$$

Equation	Solution
(iv) Current i in a circuit (L, R), t seconds after a voltage E_0 is applied to the circuit.	
$L \frac{di}{dt} + Ri = E_0$	$i = \frac{E_0}{R} (1 - e^{-Rt/L})$
(b) Solution by Integrating Factor $\frac{dy}{dx} + Py = Q$ (P, Q are functions of x only)	$y = \exp(-\int Pdx) \times [\int Q \exp(\int Pdx) dx + C]$ Integrating factor $\exp(\int Pdx)$ (This method may be applied to (iv) above)
(c) Homogeneous Equations $P \frac{dy}{dx} + Q = 0$ (P, Q homogeneous functions of x, y) Put $y = vx$ so that $dy/dx = v + x dv/dx$ and equation takes the form $x \frac{dv}{dx} = \phi(v)$	Solve as (a) above

11.2. SECOND ORDER EQUATIONS

Equation	Solution
(a) $d^2y/dx^2 = f(x)$	$y = \int \int f(x) dx + Ax + B$
(b) $d^2y/dx^2 = f(y)$	$\therefore p^2 = (dy/dx)^2$ $= 2 \int f(y) dy + A$
If $p = \frac{dy}{dx}$, $\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy} \frac{dy}{dx} = p \frac{dp}{dy}$ $\therefore p \frac{dp}{dy} = f(y)$	$\therefore \int dy / \sqrt{2 \int f(y) dy + A}$ $= \pm x + B$

Example

Inverse Square Law

$$v = dx/dt \quad \frac{1}{2}v^2 = \mu \left(\frac{1}{x} - \frac{1}{r} \right)$$

$$dx^2/dt^2 = -\mu/x^2$$

$$dx/dt = \pm \sqrt{(2\mu) \left(\frac{1}{x} - \frac{1}{r} \right)^{1/2}}$$

given $v = 0$ when $x = r$,
 $t = 0$.

Now solve as 11.1 (a)

(c) Simple Harmonic Motion

$$\frac{d^2x}{dt^2} + n^2x = 0$$

$$(dx/dt)^2 = n^2(a^2 - x^2)$$

$$(dx/dt = 0 \text{ when } x = a)$$

Multiply by integrating
factor $2dx/dt$

$$dx/dt = \pm n(a^2 - x^2)^{1/2}$$

$$x = a \cos(nt + \epsilon)$$

*Equation**Solution*

Examples

- (i) Simple Pendulum: length l

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} \sin \theta = 0 \qquad \theta = \theta_0 \cos \{[\sqrt{(g/l)}] t + \varepsilon\}$$

$$\text{i.e. } \frac{d^2\theta}{dt^2} + \frac{g}{l} \theta = 0 \text{ (approx.) Period } T = 2\pi\sqrt{(l/g)}$$

if θ is small.

- (ii) Compound Pendulum: h
and k are respectively the
distance of the centre of
gravity from the axis and
the radius of gyration
about the axis.

$$\theta = \theta_0 \cos \{[\sqrt{(gh/k^2)}] t + \varepsilon\}$$

$$\text{Period } T = 2\pi\sqrt{(k^2/gh)}$$

$$\frac{d^2\theta}{dt^2} + \frac{gh}{k^2} \sin \theta = 0$$

$$\text{i.e. } \frac{d^2\theta}{dt^2} + \frac{gh}{k^2} \theta = 0 \text{ (approx.)}$$

if θ is small.

$$(d) \frac{d^2y}{dx^2} + 2a \frac{dy}{dx} + by = 0$$

(a, b constants)

Put $y = Ae^{\lambda x}$

$$\lambda^2 + 2a\lambda + b = 0$$

(auxiliary eqn.)

$$\lambda_1, \lambda_2 \text{ are } -a \pm \sqrt{(a^2 - b)}$$

Continued on p. 56

Equation	Solution
Case I. $a^2 > b$ λ_1, λ_2, real and unequal.	$y = e^{-ax}\{A \exp [\sqrt{(a^2 - b)x}] + B \exp [-\sqrt{(a^2 - b)x}]\}$ If $b = 0$, $y = A + Be^{-2ax}$ If $a = 0$, $b = -k^2 < 0$, $y = R \cosh kx + S \sinh kx$
Case II. $a^2 = b$. λ_1, λ_2, real and equal.	$y = (R + Sx) e^{-ax}$
Case III. $a^2 - b = -p^2 < 0$. λ_1, λ_2, complex. Conjugates $-a \pm ip$.	$y = Re^{-ax} \sin (px + \varepsilon)$

11.3. LINEAR DIFFERENTIAL EQUATIONS OF ORDER n WITH CONSTANT COEFFICIENTS

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 \frac{dy}{dx} + a_0 y = 0$$

which may be written

$$(a_n D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0) y = 0 \text{ or } f(D)y = 0 \quad (i)$$

where $D = d/dx$.

Put $y = Ae^{\lambda x}$. Auxiliary equation: $f(\lambda) = 0$ which determines n values of λ so that

$$y = \sum_{r=1}^n A_r e^{\lambda_r x}$$

Example: Whirling of shafts. $\frac{d^4 y}{dx^4} - n^4 y = 0$

Auxiliary equation: $\lambda^4 - n^4 = 0$

$$y = Ae^{nx} + Be^{-nx} + R \cos nx + S \sin nx$$

11.4. PARTICULAR INTEGRALS

The general solution of

$$f(D)y = \phi(x) \quad (\text{ii})$$

is $y = u + v$ where u is the general solution of (i) the complementary function and v is a *particular integral* of (ii) given by

$$v = \frac{1}{f(D)} \phi(x)$$

$y = \text{Complementary function} + \text{Particular integral}$

Particular integrals may be found by the use of the following operational results.

$$f(D)e^{ax} = f(a)e^{ax} \quad \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$$

$$f(D^2) \left\{ \begin{array}{l} \sin \\ \cos \end{array} \right\} (ax + b) = f(-a^2) \left\{ \begin{array}{l} \sin \\ \cos \end{array} \right\} (ax + b)$$

$$\frac{1}{f(D^2)} \left\{ \begin{array}{l} \sin \\ \cos \end{array} \right\} (ax + b) = \frac{1}{f(-a^2)} \left\{ \begin{array}{l} \sin \\ \cos \end{array} \right\} (ax + b)$$

$$f(D)[e^{ax}V(x)] = e^{ax}f(D+a)V(x)$$

11.5. FURTHER SECOND ORDER EQUATIONS

Equation	Solution
$(a) \quad \frac{d^2x}{dt^2} + n^2x = c$ $(c = \text{const.})$	$x = a \sin (nt + \epsilon) + c/n^2$

Equation	Solution
(b) $\frac{d^2x}{dt^2} + n^2x = a \sin pt$ $(p \neq n)$	$x = a \sin (nt + \varepsilon)$ $+ \frac{a}{n^2 - p^2} \sin pt$
The case of resonance: $p = n$	$x = a \sin (nt + \varepsilon) - \frac{at}{2n} \cos nt$
(c) $\frac{d^2x}{dt^2} + 2a \frac{dx}{dt} + bx$ $= c \sin pt$	Complementary function given by 11.2 (d). Particular Integral is $\frac{c \sin (pt - \varepsilon)}{\sqrt{[(b - p^2)^2 + 4a^2p^2]}}$ where $\tan \varepsilon = 2ap/(b - p^2)$
(d) Homogeneous Equations $x^2 \frac{d^2y}{dx^2} + ax \frac{dy}{dx} + by = \phi(x)$ Put $x = e^t$ whence $x \frac{dy}{dx} = \frac{dy}{dt}$ $x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$ $\frac{d^2y}{dt^2} + (a - 1) \frac{dy}{dt} + by$ $= \phi(e^t)$	Solution as in 11.4.

Section 12

STATISTICS

12.1. DEFINITIONS

Statistical Data A set of measured (numerical) values of x called the *variable* or *variate* (continuous or discrete).

Population The totality of all individual values of the variable.

Sample A finite set of observed values drawn from the population.

Frequency Distribution Let the range of the variable be divided into n class intervals. Let N be the total frequency of individual values and let n_i be the frequency in the i th class, then

$$N = \sum_{i=1}^n n_i \quad \text{Relative frequency } f_i = \frac{n_i}{N} \quad \sum_{i=1}^n f_i = 1$$

Statistical data may be displayed by (a) frequency and relative frequency curves, (b) histograms, (c) frequency polygons.

Mode Value of x for which the frequency is a maximum.

Median Value of x at which the population is divided into two numerically equal parts.

Quartiles Three values of x which divide the population into four equal groups.

Percentiles Values of x which divide the population into 100 equal groups.

12.2. SAMPLE STATISTICS

Random sample of size N . Number of class intervals n .

$$r\text{th moment about the origin} = m_r' = \frac{1}{N} \sum_{i=1}^n n_i x_i^r = \sum_{i=1}^n f_i x_i^r$$

$$\text{Mean } \bar{x} = 1\text{st moment about the origin} = m_1' = \sum f_i x_i$$

$$r\text{th moment about the mean} = m_r = \frac{1}{N} \sum n_i (x_i - \bar{x})^r$$

$$= \sum f_i (x_i - \bar{x})^r$$

$$m_1 = 0 \text{ and } m_2 = s^2 = \sum f_i (x_i - \bar{x})^2 = \sum f_i x_i^2 - \bar{x}^2$$

$$m_2 = \text{Sample variance} \quad s = \text{Sample standard deviation}$$

12.3. POPULATION PARAMETERS

Similar measures for the whole population are

$$\mu = \text{population mean} \quad \sigma^2 = \text{population variance}$$

$$\sigma = \text{standard deviation.}$$

For a continuous variable x having a relative frequency or a probability density function (p.d.f) $y = f(x)$,

$$r\text{th moment about the origin} = \mu_r' = \int_{-\infty}^{+\infty} x^r f(x) dx$$

$$\text{Mean} = 1\text{st moment about the origin} = \mu_1' = \mu = \int_{-\infty}^{+\infty} x f(x) dx$$

$$r\text{th moment about the mean} = \mu_r = \int_{-\infty}^{+\infty} (x - \mu)^r f(x) dx$$

$$\begin{aligned} \text{Population variance} &= \mu_2 = \sigma^2 = \mu_2' - \mu^2 \\ &= \int_{-\infty}^{+\infty} (x - \mu)^2 f(x) dx \end{aligned}$$

12.4. ELEMENTARY PROBABILITY THEORY

An *event* comprises an *outcome* or a *group of outcomes* of an experiment. Outcomes may be represented by *sample points* in

appropriate coordinates. *Sample space* comprises the set of all such sample points.

Let p_i ($i = 1, 2, \dots, n$) be the probabilities assigned to n sample points, then $\sum_i p_i = 1$.

$$\text{Probability of Event } A = P(A) = \sum_A p_i$$

the sum being taken over the probabilities of the set A of those sample points associated with the occurrence of the event A .

Compound Probability

$P(A \cup B)$ = probability of the occurrence of the event A or B or both.

$P(A \cap B)$ = probability of the simultaneous occurrence of events A and B .

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Events A and B are *mutually exclusive* if the occurrence of one precludes that of the other, i.e. $A \cap B = \phi$. Therefore

$$P(A \cup B) = P(A) + P(B)$$

If $A_1, A_2, A_3, \dots$ are mutually exclusive events, the probability of the occurrence of at least one is $\sum_i P(A_i)$.

Conditional Probability

The probability of the occurrence of event B subject to the condition that event A has occurred is a *conditional probability*, denoted by $P(B/A)$.

$$P(B/A) = P(A \cap B)/P(A) \quad \text{or}$$

$$P(A \cap B) = P(A) P(B/A) = P(B) P(A/B)$$

If events A and B are *independent*, $P(B/A) = P(B)$ so that

$$P(A \cap B) = P(A) P(B)$$

If the events $A_1, A_2, A_3, \dots$ are independent, the probability of their simultaneous occurrence is

$$P(A_1) P(A_2) P(A_3) \dots = \prod_i P(A_i)$$

Bayes' Theorem

Suppose an event B can arise from any *one* of the initial or "causal" events $A_1, A_2, \dots, A_n$. The probability that a particular causal event A_r has resulted in the event B is given by

$$P(A_r/B) = P(A_r) P(B/A_r) / \sum_{i=1}^n P(A_i) P(B/A_i)$$

$f(x, y)$ is the *Joint Probability Density Function* of x and y if $f(x, y)$ is the probability of the simultaneous occurrence of values x and y .

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) dx dy = 1$$

If x and y are independent variables

$$f(x, y) = f_1(x) f_2(y)$$

Product Moments of a Bivariate Distribution

With respect to $x = y = 0$,

$$\mu_{rs}' = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x^r y^s f(x, y) dx dy$$

With respect to the means μ_x and μ_y of x and y ,

$$\mu_{rs} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x - \mu_x)^r (y - \mu_y)^s f(x, y) dx dy$$

If x and y are dependent (correlated) variables

$$\text{Covariance} = \text{cov}(x, y) = \mu_{11} \neq 0.$$

$$\text{Correlation coefficient} = \rho(x, y) = \mu_{11} / \sigma_x \sigma_y$$

σ_x, σ_y being the standard deviations of x and y .

12.5. VARIANCE OF A LINEAR COMBINATION OF RANDOM VARIABLES

If $z = \sum_i a_i x_i$, the mean $\bar{z} = \mu_z = \sum_i a_i \mu_i$.

$$\text{var } z = \sigma_z^2 = \sum_i a_i^2 \sigma_i^2 + \sum_i \sum_j a_i a_j \rho_{ij} \sigma_i \sigma_j$$

if the variables x_i are correlated.

$$= \sum_i a_i^2 \sigma_i^2 \text{ if the variables } x_i \text{ are independent.}$$

12.6. DISTRIBUTION OF SAMPLE MEANS

Random sample, size $n - x_i (i = 1, 2, \dots, n)$

$$\text{Sample mean} = \bar{x} = \frac{1}{n} \sum_i x_i$$

Mean of sample means $= \mu_{\bar{x}} = \mu$

Variance of sample means $= (\sigma_{\bar{x}})^2 = \sigma^2/n$

12.7. THEORETICAL FREQUENCY DISTRIBUTIONS

(a) **Binomial Distribution** (Discrete Variable)

Probability of x "successes" in n trials $= f(x) = \binom{n}{x} p^x q^{n-x}$

where p = probability of "success" and $q = (1 - p)$.

Mean $= np$. Variance $= npq$.

$$(q + p)^n = \sum_{x=0}^n \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^n f(x)$$

In N samples, each of n trials, the frequencies of 0, 1, 2, . . . , n successes are terms of the expansion of $N(q + p)^n$.

The *proportion* of successes in n trials $= z = x/n$.

$$\text{p.d.f. of } z = \binom{n}{nz} p^{nz} q^{n(1-z)}$$

$$\text{Mean of } z = \frac{1}{n} (np) = p.$$

$$\text{Variance of } z = \frac{1}{n^2} (npq) = pq/n.$$

(b) Poisson Distribution (Discrete Variable)

Derived as an approximation of the Binomial Distribution. Let $m = np$ be the mean number of successes in n trials and let n become large and $p = m/n$ small.

$$\text{Probability of } x \text{ successes} = e^{-m} m^x / x!$$

$$\text{Mean} = \text{Variance} = m$$

(c) Normal Distribution (Continuous Variable)

Denoted by $N[\mu, \sigma]$, Mean μ . St. dev. σ .

$$\text{p.d.f.} = f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left\{ \frac{-(x - \mu)^2}{2\sigma^2} \right\}$$

$$(-\infty < x < +\infty)$$

$$\text{Standard Normal Variable} = u = (x - \mu)/\sigma$$

Normal Probability Integral for the calculation of cumulative normal probabilities

$$\frac{1}{\sqrt{2\pi}} \int_0^u \exp(-\frac{1}{2}u^2) du \text{ is tabulated.}$$

12.8. ELEMENTARY SAMPLING

(a) If samples are drawn from a *normal* population, the sample means are *normally* distributed.

(b) Let x_1 be $N[\mu_1, \sigma_1]$ and x_2 be $N[\mu_2, \sigma_2]$ —a pair of *independent* variables—then $a_1x_1 \pm a_2x_2$ is

$$N[a_1\mu_1 \pm a_2\mu_2, \sqrt{(a_1^2\sigma_1^2 + a_2^2\sigma_2^2)}]$$

(c) **Central Limit Theorem** Sample size n drawn from a population of mean μ , st. dev. σ .

Sample mean $\bar{x}$ and $y = (\bar{x} - \mu)(\sqrt{n})/\sigma$
 p.d.f. of $y \rightarrow N[0, 1]$ as $n \rightarrow \infty$

(d) **Confidence Limits** for the mean μ of distribution $N[\mu, \sigma]$ from a sample, size n , mean $\bar{x}$.

$$\bar{x} \pm 1.96 \sigma/\sqrt{n} \quad (95\%) \quad \bar{x} \pm 2.58 \sigma/\sqrt{n} \quad (99\%)$$

12.9. SMALL SAMPLE DISTRIBUTIONS

(a) Small sample $y_i (i = 1, 2, \dots, n)$ drawn from population $N[0, 1]$.

$u = \chi^2 = \sum_{i=1}^n y_i^2$ has a χ^2 distribution $f(\chi^2)$ with n degrees of

freedom (d.o.f.) given by

$$f(\chi^2) = f(u) = \frac{(\chi^2)^{\frac{1}{2}n-1} e^{-\frac{1}{2}\chi^2}}{\Gamma(\frac{1}{2}n) 2^{n/2}}$$

The cumulative χ^2 distribution $\int_0^u f(u) du$ has been tabulated.

(b) **Confidence Limits** for the variance σ^2 of a normal variable. Sample, size n , variance s^2 . $(100 - \alpha)\%$ confidence limits for σ^2 are

$$ns^2/\chi_1^2 > \sigma^2 > ns^2/\chi_2^2$$

where $P(\chi^2 < \chi_1^2)$ and $P(\chi^2 > \chi_2^2)$ are each $\frac{1}{2}\alpha\%$.

(c) **F-distribution** If u, v are independently distributed χ^2 variables with ν_1, ν_2 d.o.f. then if $F = u/\nu_1 \div v/\nu_2$, the p.d.f. $p(F)$ is known and the cumulative distribution function $\int_0^F p(F) dF$ has been tabulated for various d.o.f.

(d) *F*-ratio Test

To test, on the basis of variance, whether two samples, sizes n_1 , n_2 , variances s_1^2 , s_2^2 , are likely to have been drawn from the same population of variance σ^2 .

$u = n_1 s_1^2 / \sigma^2$, $v = n_2 s_2^2 / \sigma^2$ have χ^2 distributions with $(n_1 - 1)$, $(n_2 - 1)$ d.o.f.

$$F = \frac{n_1 s_1^2}{(n_1 - 1)} \bigg/ \frac{n_2 s_2^2}{(n_2 - 1)} = \hat{\sigma}_1^2 / \hat{\sigma}_2^2$$

where $\hat{\sigma}_1^2$, $\hat{\sigma}_2^2$ are unbiased estimates of σ^2 , on the *null hypothesis* that the samples are drawn from the same population.

(e) *t*-distribution

If u is $N[0, 1]$ and v has a χ^2 distribution with ν d.o.f., the probability distribution of $t = u\sqrt{\nu}/\sqrt{v}$ has been tabulated.

Sample size n drawn from $N[\mu, \sigma]$.

Taking $u = (\bar{x} - \mu)\sqrt{n}/\sigma$ [see 12.8 (c)] and $v = ns^2/\sigma^2$ having a χ^2 distribution with $n - 1$ d.o.f.

$t = (\bar{x} - \mu)\sqrt{(n - 1)}/s$ has a t distribution with $n - 1$ d.o.f.

(f) Confidence Limits of the Mean

$$\bar{x} - \frac{s}{\sqrt{(n - 1)}} t_{\alpha/2} < \mu < \bar{x} + \frac{s}{\sqrt{(n - 1)}} t_{\alpha/2} \quad [100 \overline{1 - \alpha} \%]$$

where $P(t > t_{\alpha/2})$ is $\alpha/2$.

(g) *t*-test: Difference of Two Sample Means: Is there any essential difference between two similar samples of data? Suppose x, y are distributed $N[\mu_x, \sigma_x]$ and $N[\mu_y, \sigma_y]$ respectively. Two random samples, sizes n_x, n_y , means $\bar{x}, \bar{y}$.

On the assumption that the samples are drawn from the same population, x and y are distributed with common variance

$\sigma = \sigma_x = \sigma_y$ [justify the assumption by use of F -test, 12.9(d)]
and $\mu_x = \mu_y$.

$$t = \frac{(\bar{x} - \bar{y})}{\hat{\sigma}} \left(\sqrt{\frac{n_x n_y}{n_x + n_y}} \right) \quad \text{with } (n_x + n_y - 2) \text{ d.o.f.}$$

and $\hat{\sigma}$ is the unbiased estimate of σ from pooled data.

12.10. GOODNESS OF FIT

Population divided into k mutually exclusive classes. f_i, m_i are respectively the observed and expected frequencies in class i .

$$\sum_{i=1}^k \frac{(f_i - m_i)^2}{m_i} \text{ has a } \chi^2 \text{ distribution with } (k - 1) \text{ d.o.f.}$$

12.11 LINEAR REGRESSION

Bivariate distribution. Sample size N . Sample points: (x_i, y_i) .
Frequency f_i and $N = \sum f_i$

(a) **Least Squares Regression Line** $y = a + bx$

$$a = \bar{y} - b\bar{x} \text{ where } \bar{x} = \frac{1}{N} \sum f_i x_i, \bar{y} = \frac{1}{N} \sum f_i y_i$$

$$\text{and } b = \frac{N \sum f_i x_i y_i - \sum f_i x_i \sum f_i y_i}{N \sum f_i x_i^2 - (\sum f_i x_i)^2}$$

(b) **Unbiased Estimate of Population Correlation Ratio ρ**

is given by

$$r = s'_{xy}/s'_x s'_y = \frac{\sum f_i (x_i - \bar{x})(y_i - \bar{y})}{\{\sum f_i (x_i - \bar{x})^2 \sum f_i (y_i - \bar{y})^2\}^{1/2}}$$

where s'_x, s'_y, s'_{xy} are sample st.devs. and covariance respectively.

(c) **Significance of r** On the hypothesis that $\rho = 0$,

$r \sqrt{[(n - 2)/(1 - r^2)]}$ has a t -distribution with $(n - 2)$ d.o.f.

If the sample value of t is significant, the hypothesis is rejected and a degree of correlation is accepted.

(d) Rank Correlation (Spearman) Items ranked and assigned ordinal numbers in respect of some characteristic (usually magnitude).

Values of x have ranks u , and values of y have ranks v .

Spearman's coefficient $r = 1 - \frac{6\sum d_i^2}{(n^3 - n)}$ where $d_i = u_i - v_i$.

Section 13

NUMERICAL ANALYSIS

13.1. ROUNDING OFF ERRORS

Let A_r be the rounded-off measured value of a quantity whose exact value is A_e .

(Maximum rounding error = $\frac{1}{2}$ unit of the last decimal place in A_r)

Absolute error = $a = A_r - A_e$.

Absolute error modulus of $A_r = |a|$.

Relative error = a/A_e

Let a_1, a_2, a_3 be the absolute errors of three measured values $(A_r)_1, (A_r)_2, (A_r)_3$ respectively, then if

$$(A_r)_3 = (A_r)_1 \pm (A_r)_2 \quad |a_3| \leq |a_1| + |a_2|$$

$$(A_r)_3 = (A_r)_1 \div (A_r)_2 \quad \left| \frac{a_3}{(A_r)_3} \right| \leq \left| \frac{a_1}{(A_r)_1} \right| + \left| \frac{a_2}{(A_r)_2} \right|$$

or $(A_r)_1 \times (A_r)_2$

$$(A_r)_2 = \{(A_r)_1\}^p \quad \left| \frac{a_2}{(A_r)_2} \right| = |p| \cdot \left| \frac{a_1}{(A_r)_1} \right|$$

13.2. ERRORS IN FUNCTIONS

(a) If $u = f(x)$ and x is subject to error, then

$$\delta u = \frac{df}{dx} \delta x$$

where δu , δx are the absolute errors in u , x respectively.

(b) If $u = f(x, y, z \dots)$ then

$$\delta u = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial z} \delta z + \dots$$

13.3. TABULAR VALUES OF $f(x)$; DIFFERENCE NOTATION

Constant tabular interval = h

Value of x	Value of $f(x)$	Notation
.	.	.
.	.	.
.	.	.
x_{-2}	$f(x_0 - 2h)$	f_{-2}
x_{-1}	$f(x_0 - h)$	f_{-1}
x_0	$f(x_0)$	f_0
x_1	$f(x_0 + h)$	f_1
x_2	$f(x_0 + 2h)$	f_2
.	.	.
.	.	.
.	.	.

Forward Difference Operator Δ

First difference: $\Delta f_n = \Delta f(x_n) = f(x_n + h) - f(x_n)$

Second difference: $\Delta(\Delta f_n) = \Delta^2 f_n = \Delta f_{n+1} - \Delta f_n$

r th difference: $\Delta^r f_n = \Delta^{r-1} f_{n+1} - \Delta^{r-1} f_n$

13.4. LINEAR INTERPOLATION

If $x_p = x_0 + ph$, then $f_p = f_0 + p \Delta f_0$

13.5. SHIFT OPERATOR E

$$\begin{aligned} E\{f(x)\} &= f(x + h) & E^2\{f(x)\} &= E[E\{f(x)\}] \\ & & &= E\{f(x + h)\} = f(x + 2h) \end{aligned}$$

In general

$$E^p\{f(x)\} = f(x + ph) \quad (p \text{ +ve, -ve or fractional})$$

A more concise notation for this is

$$E^p f_n = f_{n+p}$$

$$\text{In particular, } Ef_{-1} = f_0 \quad \therefore E^{-1}f_0 = f_{-1}$$

$$E^{1/2} f_0 = f_{1/2} = f(x + \frac{1}{2}h)$$

13.6. RELATIONSHIP BETWEEN Δ AND E

$$\Delta f_n = f_{n+1} - f_n \quad \therefore (1 + \Delta)f_n = f_{n+1} = Ef_n$$

$$\therefore E \equiv 1 + \Delta \quad \text{and} \quad E^r = (1 + \Delta)^r$$

(r +ve integer)

13.7. GREGORY-NEWTON FORWARD DIFFERENCE INTERPOLATION FORMULA

Value of $f(x)$ required at $x = x_0 + ph$.

$$f_p = E^p f_0 = (1 + \Delta)^p f_0$$

$$= \left\{ f_0 + p\Delta f_0 + \frac{p(p-1)}{2!} \Delta^2 f_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 f_0 + \dots \right\}$$

13.8. BACKWARD DIFFERENCE OPERATOR ∇

Defined by $\nabla f(x_n) = f(x_n) - f(x_n - h)$ or $\nabla f_n = f_n - f_{n-1}$

$$\text{and } \nabla^r f_n = \nabla^{r-1} f_n - \nabla^{r-1} f_{n-1}$$

$$E^{-1} f_n = f_{n-1} = f_n - \nabla f_n = (1 - \nabla) f_n$$

$$E^{-1} = (1 - \nabla) \quad \text{or} \quad E = (1 - \nabla)^{-1}$$

13.9. GREGORY-NEWTON BACKWARD DIFFERENCE INTERPOLATION FORMULA

$$f_p = E^p f_0 = (1 - \nabla)^{-p} f_0 = \left\{ f_0 + p \nabla f_0 + \frac{p(p+1)}{2!} \nabla^2 f_0 + \frac{p(p+1)(p+2)}{3!} \nabla^3 f_0 + \dots \right\}$$

13.10. CENTRAL DIFFERENCE OPERATOR δ

δ is defined by $\delta f_{n+1/2} = f_{n+1} - f_n$ for all integral values of n . The operator δ may be applied to any column of the difference table so that

$$\delta^2 f_n = \delta f_{n+1/2} - \delta f_{n-1/2}$$

$$\delta^3 f_{n+1/2} = \delta^2 f_{n+1} - \delta^2 f_n \text{ etc.}$$

In particular, $\delta^{2r+1} f_{1/2} = \delta^{2r} f_1 - \delta^{2r} f_0$ ($r = 1, 2, 3, \dots$) (i)

$$\begin{aligned} \delta f_{n+1/2} &= f_{n+1} - f_n = E^{1/2} f_{n+1/2} - E^{-1/2} f_{n+1/2} \\ &= (E^{1/2} - E^{-1/2}) f_{n+1/2} \end{aligned}$$

Therefore $\delta = E^{1/2} - E^{-1/2}$

$$E^{1/2} \delta = (E - 1) = \Delta \quad \delta^2 = \Delta^2 / E = \Delta^2 / (1 + \Delta)$$

13.11. CENTRAL DIFFERENCE INTERPOLATION FORMULAE

These formulae use information placed symmetrically with respect to the interpolation interval.

(i) Bessel $f_p =$

$$\begin{aligned} f_0 + p \delta f_{1/2} + \frac{p(p-1)}{2.2!} (\delta^2 f_0 + \delta^2 f_1) + \frac{p(p-1)(p-\frac{1}{2})}{3!} \delta^3 f_{1/2} \\ + \frac{(p+1)p(p-1)(p-2)}{2.4!} (\delta^4 f_0 + \delta^4 f_1) + \dots \end{aligned}$$

(ii) Everett

Using (i) of 13.10, the odd order differences may be eliminated from the Bessel formula above giving

$$f_p = (1 - p)f_0 + pf_1 + \frac{p(p-1)(2-p)}{6} \delta^2 f_0 \\ + \frac{p(p-1)(p+1)}{6} \delta^2 f_1 + \dots$$

[Further coefficients will be found on p. 56 of *Interpolation and Allied Tables (I.A.T.)*, H.M.S.O.]

(iii) Stirling

$$f_p = f_0 + S_1(\delta f_{-1/2} + \delta f_{1/2}) + S_2 \delta^2 f_0 + S_3 (\delta^3 f_{-1/2} + \delta^3 f_{1/2}) \\ + S_4 \delta^4 f_0 + \dots$$

The coefficients $S_1, S_2, \dots$ will be found on p. 58 of *I.A.T.*

13.12. LAGRANGIAN INTERPOLATION

$f(x)$ takes the values $f_0, f_1, \dots, f_n$ for $x = x_0, x_1, \dots, x_n$ respectively. It is required to find a polynomial which takes these $(n+1)$ functional values for the same values of x . The required interpolate is found by calculating the value of this polynomial for the given value of x .

Lagrangian Interpolation Coefficients

$$L_i(x) = \frac{(x - x_0)(x - x_1) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_n)}{(x_i - x_0)(x_i - x_1) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_n)}$$

defines the $(n+1)$ coefficients for $i = 0, 1, 2, \dots, n$.

$L_i(x)$ are polynomials of degree n .

$L_i(x_j) = 0 (i \neq j)$ and $L_i(x_i) = 1$.

The **Lagrangian Polynomial** is defined by

$$L(x) = \sum_{i=0}^n L_i(x) f_i \quad \text{which is of degree } \leq n$$

Since $L(x_i) = f_i$ for $i = 0, 1, 2, \dots, n$, $L(x)$ is the required polynomial. If $f(x)$ is of degree $< n$, it may be shown that $f(x) \equiv L(x)$. If $f(x) = c$,

$$L(x) = c = \sum_{i=0}^n L_i(x) \cdot c \quad \sum_{i=0}^n L_i(x) = 1 \quad (\text{check})$$

13.13. AITKEN'S METHOD FOR INTERPOLATION WITH UNEQUAL INTERVALS

$f(x)$ has values f_0, f_1, f_j at $x = x_0, x_1, x_j$ respectively ($x_1 - x_0 \neq x_j - x_1$, in general). Determine the interpolate at x .

$$\text{First approximation} = f_{0j}(x) = \frac{\begin{vmatrix} (x - x_0) & f_0 \\ (x - x_j) & f_j \end{vmatrix}}{(x_j - x_0)}$$

$$\text{Second approximation} = f_{01j}(x) = \frac{\begin{vmatrix} (x - x_1) & f_{01}(x) \\ (x - x_j) & f_{0j}(x) \end{vmatrix}}{(x_j - x_1)}$$

13.14. NUMERICAL INTEGRATION

$$\text{If } x = x_0 + ph \quad \int_{x_0}^{x_n} f(x) dx = h \int_0^n f_p dp$$

(i) **Forward Difference Integration Formula** (derived from Gregory-Newton formula)

$$\begin{aligned} \int_{x_0}^{x_n} f(x) dx = h \left[n f_0 + \frac{n^2}{2} \Delta f_0 + \frac{n^2(2n-3)}{12} \Delta^2 f_0 \right. \\ \left. + \frac{n^2(n-2)^2}{24} \Delta^3 f_0 + \dots \right] \end{aligned}$$

and, in particular,

$$\int_{x_0}^{x_1} f(x)dx = h \left[f_0 + \frac{1}{2}\Delta f_0 - \frac{1}{12}\Delta^2 f_0 + \frac{1}{24}\Delta^3 f_0 + \dots \right]$$

Simpson's Rule (an even number n equal subdivisions of width h)

$$\int_{x_0}^{x_n} f(x)dx = \frac{1}{3}h \{ f_0 + f_n + 2(f_2 + f_4 + \dots + f_{n-2}) + 4(f_1 + f_3 + \dots + f_{n-1}) \}$$

$$\text{Estimate of error} = -\frac{h}{90} \{ \Delta^4 f_0 + \Delta^4 f_2 + \dots + \Delta^4 f_{n-2} \}$$

Trapezoidal Rule (n equal subdivisions of width h)

$$\int_{x_0}^{x_n} f(x)dx = \frac{1}{2}h \{ f_0 + 2(f_1 + f_2 + \dots + f_{n-1}) + f_n \}$$

Estimate of error =

$$-\frac{1}{12}(\mu\delta f_n - \mu\delta f_0) + \frac{11}{720}(\mu\delta^3 f_n - \mu\delta^3 f_0)$$

where μ is the averaging operator defined by

$$\mu f_p = \frac{1}{2}(f_{p+1/2} + f_{p-1/2}),$$

$$\mu\delta^r f_p = \frac{1}{2}(\delta^r f_{p+1/2} + \delta^r f_{p-1/2}) \quad (I.A.T. \text{ p. 54})$$

(ii) Central Difference Integration Formulae

(a) Derived from Bessel's interpolation formula.

$$\begin{aligned} \int_{x_0}^{x_1} f(x)dx &= h \left\{ \frac{1}{2}(f_0 + f_1) - \frac{1}{24}(\delta^2 f_0 + \delta^2 f_1) \right. \\ &\quad \left. + \frac{11}{1440}(\delta^4 f_0 + \delta^4 f_1) + \dots \right\} \\ &= h \left\{ \mu f_{1/2} - \frac{1}{12}\mu\delta^2 f_{1/2} + \frac{11}{720}\mu\delta^4 f_{1/2} + \dots \right\} \end{aligned}$$

(b) Derived from Stirling's interpolation formula.

$$\int_{x_{-1}}^{x_1} f(x)dx = 2h \left(f_0 + \frac{1}{6} \delta^2 f_0 - \frac{1}{180} \delta^4 f_0 + \dots \right)$$

13.15. NUMERICAL DIFFERENTIATION

If $x = x_0 + ph$ $f'(x) = f'(p) \frac{dp}{dx} = \frac{1}{h} f'(p)$

(i) Central Difference Formulae

(a) Derived from Bessel's Interpolation Formula.

$$f'(x) = \frac{1}{h} \left\{ \delta f_{1/2} + \frac{1}{4} (2p-1) (\delta^2 f_0 + \delta^2 f_1) + \frac{1}{8} (3p^2 - 3p + \frac{1}{2}) \delta^3 f_{1/2} \right. \\ \left. + \frac{1}{24} (2p-1)(p^2 - p - 1) (\delta^4 f_0 + \delta^4 f_1) + \dots \right\}$$

At half-way points $f'(x_0 + \frac{1}{2}h)$

$$= \frac{1}{h} \left\{ \delta f_{1/2} - \frac{1}{24} \delta^3 f_{1/2} + \frac{3}{640} \delta^5 f_{1/2} + \dots \right\}$$

(b) Derived from Stirling's Interpolation Formula.

$$f'(x) = \frac{1}{h} \left\{ \frac{1}{2} (\delta f_{-1/2} + \delta f_{1/2}) + p \delta^2 f_0 + \frac{1}{12} (3p^2 - 1) (\delta^3 f_{-1/2} \right. \\ \left. + \delta^3 f_{1/2}) + \frac{1}{12} p (2p^2 - 1) \delta^4 f_0 + \dots \right\}$$

At a tabular point x_0 , $p = 0$ so that

$$f'(x_0) = \frac{1}{h} \left\{ \frac{1}{2} (\delta f_{-1/2} + \delta f_{1/2}) - \frac{1}{12} (\delta^3 f_{-1/2} + \delta^3 f_{1/2}) \right. \\ \left. + \frac{1}{60} (\delta^5 f_{-1/2} + \delta^5 f_{1/2}) + \dots \right\} \\ = \frac{1}{h} \left\{ \mu \delta f_0 - \frac{1}{6} \mu \delta^3 f_0 + \frac{1}{30} \mu \delta^5 f_0 + \dots \right\}$$

(ii) **Forward and Backward Difference Formulae** (derived from the Gregory-Newton Formulae)

(a) Forward Difference formula.

$$f'(x) = \frac{1}{h} \left\{ \Delta f_0 + \frac{1}{2}(2p - 1)\Delta^2 f_0 + \frac{1}{6}(3p^2 - 6p + 2)\Delta^3 f_0 + \dots \right\}$$

For the tabular point x_0 , put $p = 0$ so that the Forward Difference formula becomes

$$f'(x_0) = \frac{1}{h} \left\{ \Delta f_0 - \frac{1}{2}\Delta^2 f_0 + \frac{1}{3}\Delta^3 f_0 - \frac{1}{4}\Delta^4 f_0 + \frac{1}{5}\Delta^5 f_0 - \dots \right\}$$

(b) For a point near the end of the table, use the Backward Difference formula

$$f'(x_0) = \frac{1}{h} \left\{ \nabla f_0 + \frac{1}{2}\nabla^2 f_0 + \frac{1}{3}\nabla^3 f_0 + \frac{1}{4}\nabla^4 f_0 + \frac{1}{5}\nabla^5 f_0 + \dots \right\}$$

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